Some Applications of Nonstandard Analysis to Advanced Undergraduate Mathematics

◊ Very Elementary Physics - Generalized Functions (Distributions)◊

Robert A. Herrmann

A 1991 Instructional Development Project from the
Mathematics Department
United States Naval Academy
572C Holloway Road
Annapolis, Maryland 21402-5002

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Chapter 1.

INTRODUCTION

1.1 Brief Comments

Since the time of Archimedes the major applications of infinitesimal reasoning have been in the general discipline of geometry and what we now categorize as the subject matter of Physics. All of the applications that appear in the first volume in this series – Infinitesimal Modeling – are taken from these two disciplines. The methods employed within this manual are based exclusively upon those that appear in the our major reference the *Some Applications of Nonstandard Analysis to Undergraduate Mathematics – Infinitesimal Modeling.* From time-to-time, a portion of certain applications, discussions and conclusions are directly retrieved from the Infinitesimal Modeling manual so that this Elementary Physics manual will present, from the view point of applications, a continuous and cohesive structure that parallels the standard first undergraduate physics course that requires the Calculus as a prerequisite. **One important feature of this physics manual is that many of the basic rigorous derivations are followed immediately by additional derivations that have been translated into the classical language used in most undergraduate calculus courses. This will enable most undergraduate students to more easily comprehend a derivation's logical sequence.**

1.2 Manual Structure

We will not replicate an actual physics course in this manual but rather present representative derivations using the rules established within the Infinitesimal Modeling manual for some of the more significant integral and differential equation models for the behavior of will-known natural systems.

This presentation will only be for mechanics. Obviously, we can only make a minute sampling from these very broad categories. However, it is hoped that, if care is exercised, the examples chosen will lead the instructor to seek more rigorous derivations for the more complex and refined aspects of system behavior. The individual physics instructor is certainly more intuitively and academically prepared than you author for a penetrating and rigorous investigation of the more subtle aspects of this subject.

It is the belief of your author that the most expedient approach is to train the scientific community in the rudiments of rigorous infinitesimal analysis by such devices as the Infinitesimal Modeling manual, the new infinitesimal calculus courses that have been introduced throughout the world, and manuals similar to this Physical Manual. Once individual scientists achieve a working knowledge of the basic principles then those who specialize in a given subject area are the appropriate ones to continue a more in depth exploration. What is discovered by an in depth rigorous infinitesimal approach is that simple fundamental observations lead to simple standard expressions. These expressions, after being transferred to the nonstandard model, yield a simple view of a new world called the nonstandard physical world (NSP-world). These transferred simple processes also lead to NSP-world processes that when applied within the NSP-world lead back again to standard integral or differential equation models that mirror natural system behavior. Usually, one acquires knowledge about the appropriate NSP-world processes through observation of simple or idealized natural system behavior and then accepts those NSP-world views that lead to verified predictions. It is by means of this back-and-forth approach that we gain useful knowledge about the NSP-world.

The following notation indicates the beginning and ending of each derivation. For the rigorous analytical derivation using the language of The Basic Manual, the beginning and ending are marked by a $\diamondsuit$ symbol. A second or third derivation is also be denoted by $\diamondsuit$; but, each is included within a subsection marked by {*Second Derivation*..........} or {*Third Derivation*..........}

Chapter 2.

MECHANICS

2.1. Instantaneous Velocity.

In 1686, Newton [Newton [1934]] gives what he claims is the easily comprehended notion of the "ultimate velocity," or what we now term the **instantaneous velocity**, for an actual real material object. *But by the same argument it may be alleged that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, there is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, the velocity with which the body arrives at its last place, and with which the motion ceases.* [Scholium to Lemma XI in Book 1] For the case of nonzero instantaneous velocity, one might gather from this the power of Newton's mental vision and his intuitive comprehension of future behavior. Since even though the object may not appear to move at the "instant"(i.e. an instant of time) one observes the hands of clock point at a numerical representation for the time, the object did arrive at a space location and has the capacity to change its position. It is claimed, incorrectly, that this type of change in position is noted when a second observation is made and the hands of the same clock are assumed to point at a different numerical representation for the time.

Newton's modeling of this idea is firmly rooted in his concept of the relation between geometry (the basic mathematical structure of the 1600's) and its relation to mechanics. *Geometry does not teach us to draw lines, but requires them to be drawn, for it requires that the learner should first be taught to describe these accurately before he enters geometry, then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics,....therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measure.* [Newton [1934:xvii]] Newton's claim is that our observations and intuitive comprehension of mechanics comes first in our education. These concepts are then abstracted to include the vague notion that objects have certain "capacities or potentials to do things"- the capacity or potential idea. We are told that it is after experimentation, observation and reflection that the mathematical structure is evoked and these "easy" capacity concepts are modeled.

The abstract notion of instantaneous velocity may have been "easy" for Newton to grasp, but it was incomprehensible to Berkeley and many others who believed that such abstractions could not be applied to actual real material objects. The paramount philosophy of science for Berkeley was a science of the material and directly observed universe. Any arguments that relied upon such abstractions would need to be rejected.

How can we communicate such an abstract idea to students who do not necessarily possess Newton's obvious mental ability? Tipler [1982:26] writes: *At first glance, it might seem impossible to define the velocity of a particle at a single instant, i.e., at a specific time. At a time t_1, the particle is at a single point x_1. If it is at a single point, how can it be moving? On the other hand, if it is not moving, shouldn't it stay at the same point? This is an age-old paradox, which can be resolved when we realize that to observe motion and thus define it, we must look at the position of the object at more than one time. It is then possible to define the velocity at an instant by a limiting process.* Unfortunately, Tipler has reversed Newton's original notion, that something exists prior to the modeling of motion and this something is the capacity to move. This capacity then leads to the need to seek various observations from which a numerical value can be "defined." Is it now possible to derive the well-known derivative expression for instantaneous velocity and, at least partially, retain Newton's capacity concept by infinitesimally modeling what is indeed easily observed behavior? An affirmative answer to this question depends upon your acceptance fundamental properties the Galilean theory of uniform velocity, infinitesimal analysis and its associated interpretations.

◊ Assuming that we are working in the laboratory setting with a fixed standard for the measure of (linear) distance and time, let increasing $d\colon I \to \mathbb{R}$, where $I = [a, b],\ a \neq b,\ a,\ b \in \mathbb{R}$, represent the distance an object travels over the time interval $[a, b]$. Of course, a lot has been assumed, even that it makes sense to consider time as representable by a continuum such as $[a, b]$. [*Remark.* Recent work [`Herrmann [1989]`] has shown that if time is not a continuum then there are internal functions that relate time to a continuum and these functions are infinitely close to any discrete (discontinuous) time concept.] Suppose that we extend our "observations" of finitely many cases and accept for a very simple motion that $d(t_2) - d(t_1)/(t_2 - t_1) = v = \|\vec{v}\|$ a constant (the scalar velocity), for any $t_1,\ t_2 \in [a, b],\ t_1 < t_2$. If such a motion persisted, then, of course, v can be used to calculate a change in the distance over a change in time. Let $t_1 \in (a, b)$ and Δt any positive real number such that $t_1 + \Delta t \in (a, b]$. Then, assuming a constant scalar velocity, one has that $d(t_1 + \Delta t) = d(t_1) + v\,\Delta t$ or that

$$d(t_1 + \Delta t) - d(t_1) = v\,\Delta t. \tag{1}$$

What if the distance expression was more complex than the linear type expressed by (1)? We are seeking an appropriate definition for v that extends this case. First, it follows immediately that v would be a function in $t \in [a, b]$. Since a constant function is the simplest in the collection of continuous functions, then, at this stage of our analysis, we simply require v to be continuous on $[a, b]$. However, we know from the Extreme Value Theorem that for any $[t_1, t_2] \subset [a, b], t_2 = t_1 + \Delta t,\ \Delta t > 0$, there exists $t_m,\ t_M \in [t_1, t_2]$ such that for each $t \in [t_1, t_2],\ v(t_m) \leq v(t) \leq v(t_M)$. Hence,

$$(t_2 - t_1)v(t_m) \leq (t_2 - t_1)v(t) \leq (t_2 - t_1)v(t_M), \tag{2}$$

for each $t \in [t_1, t_2]$.

Now consider the physical processes involved and correspond equation (2) to these processes. For the time span $t_2 - t_1$, it appears reasonable to state, using the case where t_m and t_M are constant and our intuitive notion of distance traveled, that the actual distance moved $d(t_2) - d(t_1)$ has the property that

$$(t_2 - t_1)v(t_m) \leq d(t_2) - d(t_1) \leq (t_2 - t_1)v(t_M), \tag{3}$$

If you accept the model for distance expressed by (3), then from the Intermediate Value Theorem there would necessarily exist some $t' \in [t_1, t_2]$ such that $d(t_2) - d(t_1) = v(t')(t_2 - t_1)$. What this means is that the distance can be calculated, knowing $v(t')$, as if it were created by a constant scalar velocity. Transfer the above results by intuitive *-transfer to the infinitesimal NSP-world (i.e. they "hold" true for the infinitesimals). Thus, for any positive $\epsilon \in \mu(0)$ and any $t_1 \in (a, b)$, it follows that there exists some $t' \in [t_1, t_1 + \epsilon]$ such that

$$({}^*d(t_1 + \epsilon) - d(t_1))/\epsilon = {}^*v(t'). \tag{4}$$

The same argument shows that if negative $\epsilon \in \mu(0)$, then there exists some $t' \in [t_1 + \epsilon, t_1]$ such that (4) as well.

Unfortunately, we do not know the value of t' in (4). But, once again, continuity of v at t_1 does allow us to write that $^*v(t') \approx v(t_1)$. Hence,

$$(^*d(t_1+\epsilon) - d(t_1))/\epsilon \approx v(t_1). \tag{5}$$

Obviously, since ϵ is an arbitrary nonzero infinitesimal then if there exists a distance function, d, that satisfies (3) for all such time intervals, then d must be differentiable at t_1 and application of the standard part operator implies that $d'(t_1) = v(t_1)$.$\diamondsuit$

{*Second derivation.* In what follows, the above derivation for the instantaneous velocity function is reworded into a quasi-classical description using slightly modified calculus terminology. The ground rules for this second derivation are:

(i) As is done in Internal Set Theory, the "*" notation is removed from the functions since whether they are nonstandard extensions of standard functions is clear from the function's argument (i.e. preimage).

(ii) The symbols "$\approx$" is translated by the term "infinitely close." This relation can be physically characterized by stating that no standard machine can measure any difference between the quantity on the left and the quantity on the right no matter how small the machine error.

(iii) Except for $\mathbb{N}_\infty$, hyperreal numbers are usually limited. Hence, simply call such a hyperreal number by the single word term "number."

(iv) We use the fact that functions defined and continuous on $[a,b]$ preserve the infinitely close concept for these numbers. That is if $t,\ t_1 \in [a,b]$ and $t \approx t_1$, then $f(t) \approx f(t_1)$. Infinitesimals may be called the "infinitely or very small." These numbers can be physically characterized as measures that are smaller than any standard machine error — measures that appear to a machine to be zero.

(v) Rather than use the standard part operator, where applicable, use the simple term "limit" in its place, since it has the same operative properties. Also use the fact that limits of two infinitely close numbers are equal.

The modified classical derivation is exactly the same until after equation (3). Then it continues as follows:

$\diamondsuit$ If you accept the model for distance expressed by (3), then from the Intermediate Value Theorem there would necessarily exist some $t' \in [t_1, t_2]$ such that $d(t_2) - d(t_1) = v(t')(t_2 - t_1)$. What this means is that the distance can be calculated, knowing $v(t')$, as if it was created by a constant scalar velocity. Now (3) and these facts hold for the infinitely small. Thus, for any positive infinitely small Δt and any $t_1 \in (a,b)$, it follows that there exists some t' such that $t_1 \leq t' \leq t_1 + \Delta t$ (i.e., $t' \in [t_1, t_1 + \Delta t]$ and

$$(d(t_1+\Delta t) - d(t_1))/\Delta t = v(t'). \tag{4}$$

The same argument shows that if Δt is a negative infinitely small number, then there is some number t' such that $t_1 + \Delta t \leq t' \leq t_1$ and once again (4) holds.

Unfortunately, we do not know the value of t' in (4). But, since $t' \approx t_1$ then continuity of v at t_1 allows us to write that $v(t') \approx v(t_1)$. Hence,

$$(d(t_1+\Delta t) - d(t_1))/\Delta t \approx v(t_1). \tag{5}$$

Obviously, since nonzero Δt is an arbitrary and infinitely small, then if there exists a distance function, d, that satisfies (3) for all such time intervals, then d must be differentiable at t_1 and the limit of the left hand side of (5), [*as Δt varies could be added, but is not necessary*] must equal the limit of the right hand side which is the constant $v(t_1)$. This implies that $d'(t_1) = v(t_1)$.$\diamondsuit$}

{*Third derivation — entirely classical.* As discussed in the Infinitesimal Modeling manual, many derivations for differential equation models require an infinitesimalizing process for behavior that is only approximated within the standard world for "small" quantities. One of the simplest illustrations of this is the differential equation model for Newton's Law of Cooling. The necessity for this special process comes from the experiential evidence that the observed behavior holds only for small measures of the independent variables and as the measures are reduced such behavior is more closely approximated by a standard functional expression. This is in direct contrast to an expression such as (3) and the discussion that follows where the results appear to hold for all intervals $[t_1, t_2]$. In the case of such concepts as the instantaneous velocity, it is possible to present an entirely classical derivation. This classical derivation begins with equation (3) and continues as follows:

$\diamondsuit$ If you accept the model for distance expressed by (3), then from the Intermediate Value Theorem there would necessarily exist some $t' \in [t_1, t_2]$ such that $d(t_2) - d(t_1) = v(t')(t_2 - t_1)$. What this means is that the distance can be calculated, knowing $v(t')$, as if it were created by a constant scalar velocity. Thus for $t_1 \in (a, b)$ and for every positive Δt such $t_2 = t_1 + \Delta t \in (a, b)$ there exists some t' such that

$$(d(t_1 + \Delta t) - d(t_1))/\Delta t) = v(t'), \tag{4}$$

and $t_1 \le t' \le t_1 + \Delta t$.

Using the Axiom of Choice, we can consider a function f defined on the respective Δt such that $f(\Delta t) = t'$. Repeating the process for the negative Δt such that $[t_1 + \Delta t, t_1] \subset (a, b)$ and extending the function f to include these negative Δt leads to the conclusion that $\lim_{\Delta t \to 0} f(\Delta t) = t_1$. From the assumed continuity of v it follows that

$$\lim_{\Delta t \to 0} \left(\frac{d(t_1 + \Delta t) - d(t_1)}{\Delta t} \right) = \lim_{\Delta t \to 0} v(f(\Delta t)) = v(t_1). \tag{5}$$

This all implies that under the conditions stated $d'(t_1) = v(t_1)$. $\diamondsuit$}

How the above derivations improve our comprehension of the concept of instantaneous velocity is discussed at the conclusion of this section. Returning to the concept of the capacity to move, Theorem 9.1.2 of the Infinitesimal Modeling manual indicates that for each $\epsilon \in \mu(0)$

$$\Delta\, {}^*\!d_{t_1}(\epsilon) = {}^*\!d(t_1 + \epsilon) - d(t_1) = v(t_1)\epsilon + \epsilon\lambda(\epsilon) = {}^*T(\epsilon) + \epsilon\lambda(\epsilon), \tag{6}$$

where λ is a local function defined by (6) and linear $T\colon \mathbb{R} \to \mathbb{R}$. Thus for each $\epsilon \in \mu(0)$, $\Delta\, {}^*\!d_{t_1}(\epsilon)$ and $v(t_1)\epsilon$ are not just infinitely close, but they are infinitely close of the first order. (See the Infinitesimal Modeling manual Section 8.3.)

Definition 2.1. (Infinitely Close of Order One). Two hyperreal valued functions defined on $\mu(0)$ are said to be **Infinitely Close of Order One or of the First Order** if for each $\epsilon \in \mu(0)$ there exists some $t \in \mu(0)$ such that

$$f(\epsilon) = g(\epsilon) + \epsilon\, t.$$

In which case, this is denoted by $f \sim_1 g$. Further, if the two functions are considered to be measuring some physical properties, then we often say that the natural world effects of these properties are **indistinguishable (at level one or on the first level)** .

It is a simple matter to show that $\sim_1$ is an equivalence relation on the set of all hyperreal valued functions defined on $\mu(0)$. The capacity to move concept is now represented in the monadic environment by noting that

$$\Delta\, {}^*d_{t_1} \sim_1 {}^*T = d'(t_1)(\cdot). \tag{7}$$

Or in words, *within the monadic world the distance represented by* $\Delta\, {}^*d_{t_1}$ *is indistinguishable from (has the same effect as) that produced by a scalar velocity* $d'(t_1)$. Of course, the function d' is termed the **instantaneous velocity**. As Newton claimed, if mechanics leads to geometry, then this is what motivated the geometric concepts of the rectifiable curve, tangents, curvature and the like that appear in the Infinitesimal Modeling manual Chapter 7, section 7.2, Chapter 8, sections 8.5 – 8.6 and Chapter 9, sections 9.5 – 9.7.

Within the foundations of any discipline it is often difficult to refine even slightly what may have been assumed previously to be an elementary and not dissectible assertion. Thus, until now, this has been with the idea of instantaneous velocity. However,

(i) let the distance function, $d(t)$, and an unknown continuous scalar velocity function, $v(t) = \|\vec{v}(t)\|$, be related by expression (3).

(ii) Let (3) hold for every time subinterval $[c,d] = [t_1 + (\Delta t)_1, t_1] \subset (a,b)$ and $[c,d] = [t_1, t_1 + (\Delta t)_2]$, where $(\Delta t)_1 < 0,\ (\Delta t)_2 > 0$.

(iii) There exists some $t' \in [c,d]$ such that the actual distance traveled $d(t_2) - d(t_1) = v(t')(t_2 - t_1)$.

Then d **must be differentiable at** t_1 **and the only standard scalar velocity function that satisfies (i), (ii) and (iii) is the function** $v(t_1) = d'(t_1)$. The instantaneous velocity is not obtained by simply postulating a definition but is derived from more elementary observations.

2.2. Acceleration

For twenty years, Galileo struggled with the problem of representing the velocity of a falling body in terms of distance `[Gillispie [1960:42]]`. After failing in every attempt, a new idea began to ferment - an idea that today seems so common place since, as illustrated in the previous section, we are taught to think of elementary velocity as expressed in a time coordinate. But, it was Galileo's revolutionary concept of time as an independent abstract entity that led to the theory of motion that Newton applied in his dynamical geometry and Mathematical Principles of Natural Philosophy [`[Newton [1686]`]. The remarkable insight exemplified by Galileo should not be underestimated. His struggle once again indicates the difficulty that scientists often face when, through reflection, they alter some well entrenched but erroneous elementary precept.

◇ As seen in the previous section, the only elementary standard function the preserves our intuitive understanding about the elementary measure of distances traveled is the instantaneous velocity. Further, in the monadic environment, the Galilean properties for constant or uniform scalar velocities and how they are compounded to yield distances traveled are indistinguishable from the actual quantities. But, now that we have accepted the instantaneous velocity, $v = \|\vec{v}\|$, as the appropriate elementary nonconstant velocity concept, we can certainly apply a section 1 type analysis to v. This requires the strict application of the Galilean theory of motion for a constant acceleration on one hand, and Newton's concept of the nonconstant acceleration (produced by a force) that leads to nonuniform velocity on the other hand.

Thus for time interval $[a,b]$ substitute in expression (3) of section 1, v for d and a representation for a continuous scalar acceleration $a = \|\vec{a}\|$ for v and obtain

$$(t_2 - t_1)a(t_m) \le v(t_2) - v(t_1) \le (t_2 - t_1)a(t_M). \tag{1}$$

The expression on the left of (1) applies the concept of a constant acceleration and measures the minimum possible [linear] uniform change in the velocity and that on the right the maximum possible uniform change. This implies, as in the case of instantaneous velocity and under parallel hypotheses as stated in section 1 (i), (ii), that there exists a unique scalar acceleration function a such that $a(t_1) = v'(t_1) = d^{(2)}(t_1)$ - the **instantaneous acceleration**. $\diamondsuit$ As with the case of the instantaneous velocity we also have

$$\Delta\, {}^*v_{t_1} \sim_1 {}^*T_a = v'(t_1)(\cdot), \tag{2}$$

which indicates that within the monadic environment that the change in velocity in the N-world is indistinguishable on the first level from that produced by a constant acceleration. However, application of Theorem 8.4.2 yields that an associated change in distance satisfies the two, difficult to visualize, statements

$$\Delta^2\, {}^*d_{t_1} \sim_1 {}^*T_a = v'(t_1)(\cdot) = d^{(2)}(\cdot), \text{ and}$$
$$\Delta^2\, {}^*d_{t_1} \sim_2 {}^*T_a = {}^*d^{(2)}(\cdot), \tag{3}$$

where $\sim_2$ is defined in the obvious manner by replacing ϵ in the last term in the equation in definition 2.1 by ϵ^2 and the second equation in (3) is the best that we can state, in generally.

2.3. Forces and Newton's "Law"

The story is told that a student asked Max Planck to explain how he perceived nuclear forces? Planck is said to have replied that he would perceive them as someone pulling on his coat sleeve. Whether or not this story is factual, experience indicates that a change in velocity is better comprehended by considering such a change as the effect of a more easily sensed "force" that may be the cause of the change. This brings use the Newton's so-called Second Law of Motion.

It is not true that Newton formulated his Second Law as it is taught in our elementary physics courses where we are told that the scalar force is proportional to the instantaneous acceleration. It is also not true that he formulated his Second Law in terms of something equivalent to the derivative of the momentum. First, Newton defined the "quantity of motion" as follows: *The quantity of motion is the measure of the same, arising from the* [scalar] *velocity and the quantity of matter conjointly* [`Newton [1934:1]`] Thus the quantity of motion is the momentum. As indicated by the explanation that follows his statement, his Second Law was the observation that *The* [uniform] *change of* [the quantity of] *motion is proportional to the* [constant] *motive force impressed; and is made in the direction of the right* [i.e., straight] *line in which the force is impressed.* [`[Newton [1934: 13]`] He proceeds in the Scholium to that section to apply this Second Law and his idea that the total effect of finitely many constant forces is additive over time to establish Galileo's discovery that the *...descent of bodies varies as the square of the time.* [`Newton [1934:21]`] In terms of a constant impulse notion, Newton's argument does not include the mass, but rather leaves the mass as the constant of proportionality. In summation notation, the argument has the following form. The time $[a, b]$ is subdivided into n equal time intervals Δt. The scalar force during these time intervals is a constant F. Then the force times the length of time impressed (i.e. the **impulse**)is proportional to the uniform change in scalar velocity, Δv, and all such uniform changes in the scalar velocity are equal in value. Hence, the composition of such forces would yield a total effect

$$\sum F(\Delta t) = F \sum \Delta t = F(b-a) \propto \sum \Delta v = v(b) - v(a) = v, \tag{1}$$

where F now becomes the *whole force*, the constant of proportionality is the same for each summand and from the additivity of velocities in a straight line motion v is the *whole velocity.* Now translating into our symbols, Newton writes $F(b-a) \propto v$ as $(b-a) \propto v$. He then states that the *spaces* [i.e., distance traveled] in *proportional times are as the product of the velocities and times;....* [`Newton [1934:21]`] Thus such a distance $d \propto v(b-a) \propto (b-a)^2$.

In his Principles, Newton does not utilize his method of fluxions, even though in other communications he does, to develop his theory of motions of material bodies. He first presents arguments delineating the "ultimate ratios" between geometric measures - arguments that employ those intuitive concepts acceptable and apparently comprehensible by geometers, expressions such as *magnitude diminished* <u>*in infinititum*</u>. He then correlates time to the measure of one leg of a right triangle and velocity to the other leg. Considering his geometric notion of the ultimate ratio of the areas of these triangles as the length of the sides *diminish* <u>*in infinititum*</u>, which he previously established was as to the squares of the *homologous sides*, he draws the conclusion that *the spaces which a body by an finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminished* [with respect to time], *are in the very beginning of the motion to each other as the squares of the times.* [`Newton [1934:34]`] In corollary *iv* he writes: *And therefore the forces are directly as the spaces described in the very beginning of the motion, and inversely as the squares of the times.* [`Newton [1934:35]`] The expression *very beginning* is used to describe the ultimate ratio concept and that this is a point force associated with an instant of time. Further notice that he has replaced the idea of constant forces over a time subinterval with forces that are being altered *continually.*

The logical vagueness of Newton's arguments can be eliminated by modern infinitesimal analysis. $\diamondsuit$ Consider the time interval $[a,b]$ and any $[t_1,t_2] \subset [a,b]$. Assume that the scalar force F is a continuous real valued function defined on $[a,b]$. We look at Newton's observations relative to constant scalar forces and there relation to uniform changes in the scalar velocity. First, from continuity, there exists $t_m,\ t_M \in [t_1,t_2]$ such that

$$F(t_m)(t_2-t_1) \leq F(t)(t_2-t_1) \leq F(t_M)(t_2-t_1) \tag{2}$$

for each $t \in [t_1,t_2]$. Let C be a constant of proportionality which, for this particular simplistic problem, is considered absolutely constant in character. Let q denote a real valued function that measures, with respect to time, the notion of the quantity of motion. Then from the actual stated Second Law it follows that the minimal possible change in momentum over the $[t_1,t_2]$ is $q(t_m) = F(t_m)(t_2-t_1)$ and the maximal change is $q(t_M) = F(t_M)(t_2-t_1)$. Now the actual velocity function in terms of time is, at present, unknown. But, whatever it may be, we consider the actual quantity of motion q to be an extension of the constant case and, hence, it is proportional, for a given object, to the actual change in velocity or to $C(v(t_2)-v(t_1))$. Assuming that such a change lies somewhere between the minimal and maximal changes then applying the continuity of the force function one obtains that there exists some $t' \in [t_1,t_2]$ such that

$$F(t')(t_2-t_1) = C(v(t_2)-v(t_1)). \tag{3}$$

Letting $t_1 \in (a,b)$ and noticing that the above is assumed to hold for *any* subinterval $[t_1, t_1 + \Delta t],\ \Delta t > 0$ of $(a,b]$ or *any* subinterval $[t_1+\Delta t, t_1],\ \Delta t < 0$ of $[a,b)$ then *-transfer yields that for nonzero $\epsilon \in \mu(0)$ there exists some $t' \approx t_1$ such that

$$^*F(t') = \frac{C(\,^*v(t_1+\epsilon)-v(t_1))}{\epsilon} \approx F(t_1). \tag{4}$$

Taking the standard part operator we arrive at a derivation of our modern Newton's Second Law of Motion. That for all of this to occur there must exist a velocity function that is differentiable at t_1 and the only relation between such a point force function and such a velocity function is

$$F(t_1) = Cv'(t_1) = Ca(t_1), \tag{5}$$

where a is the **instantaneous acceleration.** $\diamondsuit$ As previously, this can be further related to the change in momentum for an infinitesimal time by the expression

$$C(\Delta\,{}^*v) \sim_1 F(t_1)(\cdot). \tag{6}$$

Or, as stated in words, that the change in momentum over an infinitesimal time is indistinguishable on the first level from that produced by a constant force applied to infinitesimal time periods. Moreover, the easily grasped concept of the impulse was the starting point in Newton's original arguments. Unfortunately, some modern textbooks do not introduce the impulse relative to constant forces as a first principle.

{*Second derivation.* Start the modification of the above after equation (3).

$\diamondsuit$ Letting $t_1 \in (a, b)$ and noticing that the above is assumed to hold for *any* subinterval $[t_1, t_2]$ of $[a, b]$ then all of the above holds for infinitely small subintervals. Thus for nonzero infinitely small Δt there exists some $t' \approx t_1$ such that

$$F(t') = \frac{C(v(t_1 + \Delta t) - v(t_1))}{\Delta t} \approx F(t_1). \tag{4}$$

Taking the limit of the left hand side [as Δt varies] we arrive at a derivation of our modern Newton's Second Law of Motion.

$$C \lim \frac{(v(t_1 + \Delta t) - v(t_1))}{\Delta t} = F(t_1). \tag{5}$$

Thus for all of this to occur there must exist a velocity function that is differentiable at t_1 and the only relation between such a point force function and such a velocity function is

$$F(t_1) = Cv'(t_1) = Ca(t_1), \tag{6}$$

where a is the **instantaneous acceleration.** $\diamondsuit$}

The reader may replicate the third derivation along with the discussion of properties (i), (ii) and (iii) as they appear in Section 2.1 for the instantaneous acceleration and Newton's Second Law.

Notice that one derivation method – the maximum and minimum method – yields the instantaneous velocity, instantaneous acceleration and the modern Second Law of Motion from what may be more fundamental observations.

2.4. Vectors

For constant forces, Newton's Corollary I to his three laws establishes for constant forces the idea that bodies move in $\mathbb{R}^2$ by the vector composition of two scalar forces acting simultaneously upon a particle or fixed point in a material body. [`Newton [1934:14]`] Newtonian mechanics may appear to begin with the idea that the position of a particle in an n-dimensional coordinate system is dependent upon the composition of n forces (the cause); but, in actually, it is the position (i.e. the effect) that is the fundamental concept. The position of a particle is defined in terms of n coordinate functions, each expressed in the independent parameter — time. This leads to the **position vector (i.e. radius vector)** $\vec{r} = (x_1(t), \dots, x_n(t)),\ t \in [a,b]$. Applying the previous section to these coordinate functions independently, leads to the vector form for the instantaneous velocity, instantaneous acceleration, and force.

In the Infinitesimal Modeling manual, the geometric concept of the length of a continuous curve is fully discussed and, where possible, this length is correlated to the integral by means of our integral rules. Following Newton's notion of a dynamic geometry, the same conclusions evolve from the instantaneous velocity. Suppose that a point force $\vec{F}(t),\ t \in [a,b]$, is the cause that induces an instantaneous acceleration $\vec{a}(t)$ upon a particle. Since $\vec{F}$ was assumed continuous on some time interval $[a,b]$ then $\vec{v}\,'$ is continuous on $[a,b]$. Consequently, $\vec{v}$ and $\|\vec{v}\|$ are continuous on $[a,b]$. Assuming we are in $\mathbb{R}^n$ then, noting that for $t \in [a,b],\ \|v(t)\| = \sqrt{(x_1'(t))^2 + \cdots (x_n'(t))^2} = d'(t)/dt$, we obtain

$$d(b) - d(a) = \int_a^b \sqrt{(x_1'(t))^2 + \cdots (x_n'(t))^2}\, dt, \tag{1}$$

which is the same result obtained in the Infinitesimal Modeling manual by considering hyperpolygonal representations for the length of a continuously differentiable geometric curve.

Let the path of motion be represented by continuous $c\colon [a,b] \to \mathbb{R}^n$, where $c(t) = (x_1(t), \dots, x_n(t))$. By Theorems 9.5.1 and 9.5.2 of Chapter 9 and Theorem N.1 at the end of Chapter 10 in the Infinitesimal Modeling manual it is shown that if $\vec{v}(t) = \vec{c}\,'(t) \neq \vec{0},\ t \in (a,b)$, then the unit tangent $\vec{T} = \pm\vec{c}\,'(t)/\|\vec{c}\,'(t)\|$ and that $\vec{T}$ is almost parallel to every hyperpolygonal directed line segment $({}^*c(t+dx) - c(t))/\|{}^*c(t+dx) - c(t)\|$ for each nonzero $dx \in \mu(0)$. This implies that the unit instantaneous velocity vector $\vec{v}(t)/\|\vec{v}(t)\|$ is not only one of the two possible unit tangent vectors but also has the property of being almost parallel to each $({}^*c(t+dx) - c(t))/\|{}^*c(t+dx) - c(t)\|$. Since the effect of vectors as a model for natural world (i.e. N-world) behavior is often related to the physical concept of "direction" then as far as the N-world effects are concerned the direction of the standard velocity vector attached to the point $c(t)$ is indistinguishable from the direction of $\pm({}^*c(t+dx) - c(t)),\ 0 \neq dx \in \mu(0)$. The indistinguishableness of such effects is beneficial when energy is to be considered.

2.5. Energy and Force Fields

In the Infinitesimal Modeling manual, the line integral is employed to measure the *energy expended within a forced field while moving along a curve* (Application 8.2.1) and the customary method of line integral evaluation obtained. We examine here the first portion of this derivation.

♢ In experimental physics, the concept of "work" (energy expended) is introduce. **All one needs to do is to establish its properties for a polygonal curve.** Suppose we have continuous force field $F\colon E \to \mathbb{R}^n$. Let $\mathcal{P}_k \subset \mathbb{R}^n$ be any finite polygonal curve, ℓ_j one of the line segment portions of $\mathcal{P}_k$ with $\vec{v}_j$ denoting this line segment considered as a directed line segment in the direction of motion through the field F. If F is constant on ℓ_j, then the work done moving along ℓ_j is defined as $W(\ell_j) = F \bullet (\vec{v}_j/\|\vec{v}_j\|)\|\vec{v}_j\|$, where length of $\ell_j = \|\vec{v}_j\|$. What if the force field is not constant? Consider $\mathcal{P}_k$ as represented by a continuous $\ell\colon [a,b] \to \mathbb{R}^n$ and assume that F is defined on ℓ. Then for a given $\ell_j = \{(x_1(t), \dots, x_n(t)) \mid t \in [t_{j-1}, t_j]\}$ there exists some $t_m,\ t_M$ such that $W_m(\ell_j) = F(\ell(t_n)) \bullet \vec{v}_j \leq W(\ell_j) = F(\ell(t)) \bullet \vec{v}_j = W_M(\ell_j) = F(\ell(t_M)) \bullet \vec{v}_j$ for each $t \in [t_{j-1}, t_j]$. Let's make the one assumption that the actual amount of work expended moving along the line segment ℓ_j is $W(\ell_j)$ and that $W_m(\ell_j) \le W(\ell_j) \le W_M(\ell_j)$. Then from continuity there exists some $h'_j \in [t_{j-1}, t_j]$ such that $W(\ell_j) = F(\ell(h'_j)) \bullet \vec{v}_j$. The idea of the nonconstant force field over a line segment is embedded into the the NSP-world by *-transfer assuming that what has been established above holds for all such polygonal curves. Hence, let $\mathcal{P}_\Omega$ be a hyperpolygonal representation for the curve generated by a fine partition, ℓ_j an hyperline segment in $\mathcal{P}_\Omega$. Since F is continuous on E then ${}^*F$ is defined on $\mathcal{P}_\Omega$. It follows that ${}^*W(\ell_j) = {}^*F(\ell_j(h'_j)) \bullet {}^*\vec{v}_j$. For polygonal curves, in general, the work done is an additive function. Thus for the hyperpolygonal curve $\mathcal{P}_\Omega$

$$ {}^*W(\mathcal{P}_\Omega) = \sum_{j=1}^{\Omega} {}^*F(\ell_j(h'_j)) \bullet {}^*\vec{v}_j. \tag{1} $$

The following is established within this derivation and in the Infinitesimal Modeling manual. Let $c\colon [a,b] \to \mathbb{R}^n$ be a continuous differentiable curve with graph C and assume that $c'(t) \ne 0$ for each $t \in [a,b]$. Assume that uniformly continuous $F\colon E \to \mathbb{R}^n$, open $E \supset C$. The work done in moving through the force field on the path C is

$$ W(C) = \mathtt{st}\left(\sum_{j=1}^{\Gamma} {}^*F(\ell_j(h'_j)) \bullet {}^*\vec{v}_j\right) = \int_C F \bullet d\vec{R}, \tag{2} $$

where the integral is the line integral over c and $\mathcal{P}_\Gamma$ is <u>any</u> hyperpolygonal representation for the curve C. Thus, under the hypotheses given, the work done in the NSP-world moving along these hyperpolygonal curves is indistinguishable, in general, from what is accepted as the N-world work in traveling along the curve. ♢

{*Second derivation.* ♢ See the second derivation for the general impulse in section 2.6 and modify the above accordingly. ♢}

It is significant to realize that even though the line integral will exist under less constraints and one could extend the concept of energy, to say curves that are rectifiable but not smooth, this need not imply that there is a meaningful correlation between this extended concept and its simplistic restriction to polygonal curves. On the other hand, the above derivation once again utilizes a maximum and minimum approach relative to a basic geometric element.

2.6 <u>General Impulse</u>

The energy or work done is considered to be the standard part of any one of a collection of hyperfinite sums each term of which is modeled by the amount of energy expended moving along a hyperline segment through a constant force field. The hyperfinite sums may be manipulated internally as if they were finite sums and preserve the intuitive concept of finite additivity. Unfortunately, when an object is represented by a simple differential equation obtained by such methods as illustrated in section 2.1, 2.2, and 2.3 then many authors solve such expressions by elementary differential equations methods without given full infinitesimal meaning to the process involved. The Fundamental Theorem of Integral Calculus is relevant to the existence of such solutions; but, it seems, that in order to grasp the NSP-world significance of the concept being measured that due consideration should be given to the integral solution rather than simply expressing the result by means of a rote procedure. As an illustration of this consider the extension of the concept of the impulse.

Previously the impulse was a natural notion restricted to constant force fields. A scalar force F, when applied for a period of time Δt produces an altered momentum Δp. This leads to the expression $F\Delta t = \Delta p$. Then $F\Delta t = I$ is defined as the impulse. The extension to vector notation is obvious $\vec{I} = \vec{F}\,\Delta t = \Delta\vec{p}$, with a meaningful measure being the Euclidean norm. Applying the infinitesimalizing process (2), (3), (4) and (5) of section 2.3, where $C(v(t_2) - v(t_1))$ is replaced by $\vec{p}(t_2) - \vec{p}(t_1)$ leads to the customary elementary derivative expression

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{1}$$

Of course, the vector $\vec{F} = (p'_1, \ldots, p'_n)$ where the p_i are the components for the momentum vector $\vec{p}$.

As far as a generalization for the impulse is consider there are various approaches. The Self-evident Max. and Min. Theorem 6.2.4 in the Infinitesimal Modeling manual applied to the components of $\vec{p}$ leads to the conclusion, if one has an intuitive comprehension of the basic additive properties for the impulse concept, that the proper expression for the general impulse for a continuous time dependent force field $\vec{F}$ over the time interval $[a, b]$ should be $\vec{I} = \int_a^b \vec{F}\,dt$. However, it may be more motivational and instructive to consider, once again, hyperpolygonal paths of motion since momentum is modeled after the motion concept and the concept of hyperfinite summation only. The summation of the measures of elementary physical processes still remains a primary modeling procedure that dictates the overall physical effect.

$\diamondsuit$ Suppose, as in section 2.5, that uniformly continuous force field $F\colon E \to \mathbb{R}^n$, open $E \subset \mathbb{R}^n$, $F = (F_1, \ldots, F_n)$. For a line segment $\ell' \subset E$, determine by a linear system of equations over the time interval $[t_1, t_2]$ and for a component F_i equation (2) section 2.3 can be re-expressed in terms of an impulse vector $\vec{I} = (I_1, \ldots, I_n)$ as $F_i(t_m)(t_2 - t_1) = I_{im} \leq F_i(t)(t_2 - t_1) \leq F_i(t_M)(t_2 - t_1) = I_{iM}$. Thus there is some $t'_2 \in [t_1, t_2]$ such that the actual i'th component of the impulse over that line segment is $I_i(t'_2) = F_i(t'_2)(t_2 - t_1)$. By *-transfer this holds for any hyperline segment. **Over a finite polygonal line ℓ it is assumed that the total i'th component of the impulse is the simple sum of the i'th component over the individual line segments**. Considering the impulse to be a function defined over the set of all finite polygonal curves in E, then for any hyperpolygonal curve $\mathcal{P}_\Gamma$

$$^*I_i(\mathcal{P}_\Gamma) = \sum_{j=1}^{\Gamma} {^*F_i}(t'_j)(t_j - t_{j-1}), \tag{2}$$

where $\{t_0, \ldots, t_\Gamma\}$ is the fine partition of $^*[a, b]$ that generates $\mathcal{P}_\Gamma$.

Now from uniform continuity, $^*F_i(t'_j) = {^*F_i}(t_j) + \epsilon_j,\ \epsilon_j \in \mu(0)$. Since any finite set of real numbers contains a maximum, then any hyperfinite set of hyperreal numbers contains a maximum. Thus there exists some $\epsilon = \max\{\epsilon_1, \ldots, \epsilon_\Gamma\} \in \mu(0)$ and following the elemental derivation process from the Infinitesimal Modeling manual, section 8, we have

$$\left|\sum_{j=1}^{\Gamma} \epsilon_j(t_j - t_{j-1})\right| \leq \sum_{j=1}^{\Gamma} |\epsilon_j|\,(t_j - t_{j-1}) \leq |\epsilon| \sum_{j=1}^{\Gamma}(t_j - t_{j-1}) = |\epsilon|\,(b - a) \Rightarrow$$

$$\lambda_i = \sum_{j=1}^{\Gamma} \epsilon_j(t_j - t_{j-1}) \in \mu(0). \tag{3}$$

Substitution into equation (2) yields

$$^*I_i(\mathcal{P}_\Gamma) = \left(\sum_{j=1}^{\Gamma} {^*F_i}(t_j)(t_j - t_{j-1})\right) + \lambda_i, \tag{4}$$

Now F_i is continuous. Thus $\mathtt{st}({}^*I_i) = \mathtt{st}(\sum_{j=1}^{\Gamma} {}^*F_i(t_j)(t_j - t_{j-1}))$ exists, by Theorem 5.1.2 of the Infinitesimal Modeling manual, since $\{t_1, \ldots, t_\Gamma\}$ is an internal intermediate partition and $\mathtt{st}({}^*I_i) = \int_a^b F_i(t)\,dt$. Notice that the hyperpolygonal curve was only an auxiliary notion since this last result is determine by a fine partition of ${}^*[a,b]$ and is, indeed, the same for all such fine partitions. Extending the impulse to the vector $\vec{I}$, then $\vec{I} = \mathtt{st}({}^*\vec{I}) = \int_a^b \vec{F}(t)\,dt.$◇

{*Second derivation.* First, it should be obvious that there is no complete classical counterpart to the nonstandard language used in an elemental derivation. Recall, however, how it is possible to us a quasi-classical language in the second derivation given in section 2.1. (i) As is done in Internal Set Theory, the "*" notation is removed from the functions since whether they are nonstandard extensions of standard functions is clear from the function's argument (i.e. preimage).

(ii) The symbols "$\approx$" is translated by the term "infinitely close." This relation can be physically characterized by stating that no standard machine can measure any difference between the quantity on the left and the quantity on the right no matter how small the machine error.

(iii) Except for $\mathbb{N}_\infty$, hyperreal numbers are usually limited. Hence, simply call such a hyperreal number by the single word term "number."

(iv) We use the fact that functions defined and continuous on $[a, b]$ preserve the infinitely close concept for these numbers. That is if $t,\ t_1 \in [a, b]$ and $t \approx t_1$, then $f(t) \approx f(t_1)$. Infinitesimals may be called the "infinitely or very small." These numbers can be physically characterized as measures that are smaller than any standard machine error — measures that appear to a machine to be zero.

(v) Rather than use the standard part operator, where applicable, use the simple term "limit" in its place, since it has the same operative properties. Also use the fact that limits of two infinitely close numbers are equal.

For the elemental derivation method some additions to this quasi-classical language are necessary. These additions will necessarily be somewhat less precise for the concepts of the first-order property and the internal object will not be mentioned.

(vi) Let the region J be any of those studied in elementary calculus.

(vii) Hyperline segments are termed "infinitely small line segments," which means line segments of infinitesimal length.

(viii) Hyperpolygonal curves may be called "infinitesimal polygonal curves" and defined as "polygonal curves with each line segment being an infinitely small segment."

(ix) Call a fine partition of an interval an "infinitesimal partition". This means that it has a $\Gamma + 1$ number of elements that determine Γ subintervals each of which is infinitely small in length.

(x) Hyperfinite sums are termed as "special finite sums" and behave as if they are finite sums.

(xi) Hyperfinite sets are termed as "special finite sets" and they also behave as if they are finite.

(xii) For the special finite sets, or special finite sums, Γ denotes the "number" of elements or terms, respectively.

(xiii) The product of an infinitesimal and a number [assuming limited] is an infinitesimal.

(xiv) Any nonnegative number less than or equal to an infinitesimal is an infinitesimal.

◇ Suppose, as in section 2.5, that uniformly continuous force field $F\colon E \to \mathbb{R}^n$, open $E \subset \mathbb{R}^n$, $F = (F_1, \ldots, F_n)$. For a line segment $\ell' \subset E$, determine by a linear system of equations over the time interval $[t_1, t_2]$ and for a component F_i equation (2) section 2.3 can be re-expressed in terms of an impulse vector $\vec{I} = (I_1, \ldots, I_n)$ as $F_i(t_m)(t_2 - t_1) = I_{im} \leq F_i(t)(t_2 - t_1) \leq F_i(t_M)(t_2 - t_1) = I_{iM}$. Thus there is some $t'_2 \in [t_1, t_2]$ such that the actual i'th component of the impulse over that line segment is $I_i(t'_2) = F_i(t'_2)(t_2 - t_1)$. This result holds for any infinitesimal line segment. **Over a finite polygonal line ℓ it is assumed that the total i'th component of the impulse is the simple**

sum of the i'th component over the individual line segments. Considering the impulse to be a function defined over the set of all finite polygonal curves in E, then this i'th component of the impulse is defined on an infinitesimal polygonal curve $\mathcal{P}_\Gamma$ with Γ many sides and its value is the special finite sum

$$I_i(\mathcal{P}_\Gamma) = \sum_{j=1}^{\Gamma} F_i(t'_j)(t_j - t_{j-1}), \tag{2}$$

where $\{t_0, \ldots, t_\Gamma\}$ is an infinitesimal partition of $[a, b]$ with infinitely small subintervals that generates $\mathcal{P}_\Gamma$.

Now since the above results hold for these subintervals and $t' \in [t_{j-1}, t_j]$ for $1 \leq j \leq \Gamma$ then $t' \approx t_j$. Hence, $F_i(t'_j) \approx F_i(t_j)$ and from the definition of $\approx$ for each j there exists some infinitely small ϵ_j such that $F_i(t'_j) = F_i(t_j) + \epsilon_j$. Since any finite set of real numbers contains a maximum, then any special finite set of numbers contains a maximum. But, $\{\epsilon_1, \ldots, \epsilon_\Gamma\}$ is a special finite set. Thus there exists some infinitely small $\epsilon = \max\{\epsilon_1, \ldots, \epsilon_\Gamma\}$. Recall that what holds true for finite sums holds true for special finite sums. Thus the triangle inequality holds. Consequently

$$\left|\sum_{j=1}^{\Gamma} \epsilon_j(t_j - t_{j-1})\right| \leq \sum_{j=1}^{\Gamma} |\epsilon_j|\,(t_j - t_{j-1}) \leq |\epsilon| \sum_{j=1}^{\Gamma}(t_j - t_{j-1}) = |\epsilon|\,(b - a)$$

But, since $|\epsilon|\,(b - a)$ is infinitely small then

$$\lambda_i = \sum_{j=1}^{\Gamma} \epsilon_j(t_j - t_{j-1}) \tag{3}$$

is infinitely small. Substitution into equation (2) and again using the fact that a special finite sum behaves like a finite sum yields

$$I_i(\mathcal{P}_\Gamma) = \left(\sum_{j=1}^{\Gamma}(F_i(t_j) + \epsilon_j)(t_j - t_{j-1})\right) =$$

$$\left(\sum_{j=1}^{\Gamma} F_i(t_j)(t_j - t_{j-1})\right) + \lambda_i, \tag{4}$$

Hence,

$$I_i(\mathcal{P}_\Gamma) \approx \left(\sum_{j=1}^{\Gamma} F_i(t_j)(t_j - t_{j-1})\right). \tag{5}$$

Now F_i being continuous implies from the definition of the integral that $\lim \left(\sum_{j=1}^{\Gamma} F_i(t_j)(t_j - t_{j-1})\right)$ $= \int_a^b F_i(t)\,dt$. Taking the limit of expression (5) yields that $I_i = \int_a^b F_i(t)\,dt$. Extending the impulse to the vector $\vec{I}$, then $\vec{I} = \lim \vec{I} = \int_a^b \vec{F}(t)\,dt$. $\diamondsuit\}$

There are, at least, two ways used to arrive at the relation between the change of momentum $\vec{p}(b) - \vec{p}(a)$ and the impulse $\vec{I}$. The first is the usual one of simply stating that $\vec{I} = \int_a^b \vec{F}(t)\,dt = \int_a^b (d\vec{p}/dt)\,dt = \vec{p}(b) - \vec{p}(a)$ from the Fundamental Theorem of Integral Calculus. There is a second method utilizing hyperfinite summation that incorporates the fact that F_i may be considered as uniformly continuous on $[a, b]$. Since this notion is significant for the proper intuitive understanding of the underlying physical processes, we give an illustration using the momentum vector $\vec{p} = (p_1, \ldots, p_n)$.

$\diamondsuit$ Start with equation (2) and substitute ${}^*p'_i(t'_j)$ for $F_i(t'_j)$. This obtains

$$ {}^*I_i(\mathcal{P}_\Gamma) = \sum_{j=1}^{\Gamma} {}^*p'_i(t'_j)(t_j - t_{j-1}), \tag{5} $$

Then by the nonstandard mean value theorem there exists some $h'_j \in (t_{j-1}, t_j)$ such that

$$ {}^*p_i(t_j) - {}^*p_i(t_{j-1}) = {}^*p'_i(h'_j)(t_j - t_{j-1}). \tag{6} $$

But uniform continuity of p'_i on $[a, b]$ implies that there exists some $\delta_j \in \mu(0)$ such that ${}^*p'_i(h'_j) = {}^*p'_i(t'_j) + \delta'_j$. The elemental derivation process then yields

$$ \delta_i + \sum_{j=1}^{\Gamma} {}^*p'_i(h'_j)(t_j - t_{j-1}) = \sum_{j=1}^{\Gamma} p_i(t'_j)(t_j - t_{j-1}). \tag{7} $$

Substituting (6) into (7) and again using the elemental derivation process yields

$$ \delta_i + p_i(b) - p_i(a) = \sum_{j=1}^{\Gamma} {}^*p'_i(t'_j)(t_j - t_{j-1}) = {}^*I_I(\mathcal{P}_\Gamma),\ \delta_i, \lambda_i \in \mu(0). \tag{8} $$

Consequently, $p_i(b) - p_i(a) = \mathtt{st}({}^*I_i(\mathcal{P}_\Gamma))$ implies the result sought that $\vec{p}(b) - \vec{p}(a) = \mathtt{st}({}^*\vec{I}(\mathcal{P}_\Gamma)) = \vec{I}$.$\diamondsuit$

{*Second derivation.* $\diamondsuit$ Modify the second derivation for the general impulse integral.}$\diamondsuit$

Once again, the above illustrates that the total impulse is not dependent upon the hyperpolygonal path along which the object moves, but is simply the change in momentum. However, by physical intuition, an object has changed its momentum by traversing a physical path C and we are using a hyperpolygonal representation for such a path of motion. For any two such hyperpolygonal curves $\mathcal{P}_\Gamma,\ \mathcal{P}_\Lambda$, it follows from the hypotheses that ${}^*\vec{I}(\mathcal{P}_\Gamma) \approx {}^*\vec{I}(\mathcal{P}_\Lambda)$. Now the basic geometric measure for C, the length, is from the derivation in the Infinitesimal Modeling manual completely determined by the *-length of each of these hyperpolygonal representations. Therefore, it seems appropriate to consider the standard impulse over the curve C to be the unique effect $\mathtt{st}({}^*\vec{I}(\mathcal{P}_\Gamma))$, where $\mathcal{P}_\Gamma$ is any hyperpolygonal representation for C.

Why does the elemental derivation process work and what is it indicating about integral styled quantities? All of the basic quantities in any expression prior to considering their hyperfinite sum must not only be infinitely close but must be, in the above case, infinitely close of order one. The hyperfinite summation of infinitesimals need not be infinitesimal or even limited; but, depending on the type of limited objects this type of special infinite closeness utilizes, then the elemental derivation process does imply that this particular hyperfinite sum of infinitesimals is infinitesimal. In order to guarantee that this is the case, strong hypotheses where required for the functions involved. Except for the possibility of restricting the hyperfinite summation to special sets of partitions, such as in the case of the gauge type integral discussed in section 8.9 of the Infinitesimal Modeling manual, it, at present, has not been possible to obtain rigorous derivations for integral expressions without these strong hypotheses.

Chapter 3.

SLIGHTLY LESS BASIC MECHANICS

3.1 Mass

Since the Infinite Sum Theorems, the rules IR1–IR6 and the Self-Evident Theorems that appear in the Infinitesimal Modeling manual, have not been known to the physicist previously then almost all of the elementary definitions or derivations that involve infinitesimal quantities and the integral have relied upon elemental methods. These methods refer directly to the "particle", point charges, point masses and other such notions. This process obviously forces the measure to have the same properties as the integral and the converse of starting with a functional, considering fundamental properties and showing that such a functional must be a measure as defined by some integral need not be considered. The mathematician might find this converse approach as appealing. But, it is more of a global view for a particular scenario and elementary instruction in the physical sciences tends to force upon a student an atomistic view in the sense that complex observed behavior is conceived of as the effects produce by minuscule objects. This is the prevalent textbook approach.

The Self-Evident Theorems that appear in the Infinitesimal Modeling manual require, at least, three global assumptions as well as a strong additivity property for the measure under consideration. The simplifications for regions that have Jordan-content, Propositions 6.7 and 6.10 as they appear in appendix 6, are very easy to apply in the laboratory setting since one need only investigate these properties on a rectangular interior to the region. However, this does require the assumption that the measure being considered is Jordan-like. Theorem 7.2.2 in the Infinitesimal Modeling manual implies that the integral over a Jordan-measurable region is an ordinary Jordan-like measure. Further, all of the customary geometric regions used within elementary physics are all compact Jordan-measurable sets and these simplified self-evident theorems may be applied. But the technical difficulties of establishing these simplified self-evident theorems should not be underestimated as amply illustrated by the formal proofs that appear in appendix 6 of the Infinitesimal Modeling manual. These proof methods are well beyond almost all undergraduates who require even advance physics courses as part of their curriculum.

In the Infinitesimal Modeling manual the Infinite Sum Theorem is the only approach used to obtain the mass of an object as determined by a continuous density function and this approach does not appear in any present day physics textbook. Consequently, it is obvious that, for the present, the elemental method with the notion of hyperfinite summation is more appropriate. As discussed towards the end of section 2.6, unless we restrict integral modeling to rectangular regions then, even within the elemental derivation, it is necessary to assume that the quantity being measured is Jordan-like. A brief review of this concept is useful.

Let R be an n-dimensional rectangle and $R \subset \mathbb{R}^n$. Suppose that Jordan-measurable $J \subset R \subset \mathbb{R}^n$, where to avoid trivialities we always assume that the interior of J is nonempty. Let $\mathcal{P}$ be a set of partitions of R, where, as usual, if $P \in \mathcal{P}$, then each $S \in P$ is but an n-dimensional subrectangle in R. Define on $\mathcal{P}$ the following map "inn." For each $P \in \mathcal{P}$, let $\text{inn}(P) = \{S \mid S \in P \text{ and } S \subset J\}$. Of course, the set $\text{inn}(P)$ is called a set of **inner subrectangles.** Let $I_{\mathcal{P}} = \{\cup A \mid \exists P(P \in \mathcal{P} \wedge A = \text{inn}(P))\} \bigcup \{S \mid \exists P(P \in \mathcal{P} \wedge S \in \text{inn}(P))\}$. When considering the nonstandard extension of inn to a fine partition Q we usually do not write this as ${}^*\text{inn}(Q)$ but rather retain the original notation $\text{inn}(Q)$. Since any fine partition Q is hyperfinite and the set $\text{inn}(Q)$ is an internal subset of Q then it is also hyperfinite. For simplicity, as far as the functional (i.e. measure) $B\colon \{I_{\mathcal{P}}, J\} \to \mathbb{R}$ is concerned, when the elemental derivation method is used it is assumed that B is, at least, simply additive on each $\mathrm{inn}(P)$, when $P \in \mathcal{P}$. This means that if $P \in \mathcal{P}$ and $A = \mathrm{inn}(P)$, then $B(\cup A) = \sum_{S\in A} B(S)$.

In general, if a generating function for the above functional B is constant over the entire Jordan-measurable J, then it is necessary that B display certain Jordan-measure properties. In particular, if v denotes the Jordan-measure, then for any fine parition Q of ${}^*R$, it follows from Theorem 7.2.2 that $v(J) = \mathtt{st}(\sum_{S\in A} {}^*v(S)) = \mathtt{st}({}^*v(\cup S))$. Thus, $v(J) \approx {}^*v(\cup A)$. Of course, ${}^*v(S)$ is but the product of the infinitesimal lengths of the sides of the n-dimensional subrectangle S. Bearing in mind the basic properties of Jordan-measure, we assume that for a fine partition Q that the functional B has the property that $B(J) \approx {}^*B(\cup A)$, where $A = \mathrm{inn}(Q)$. This property is termed the **Jordan-like** property and a standard criterion for the functional B to be Jordan-like for all fine partitions is given in the Infinitesimal Modeling manual. Of course, these technical matters need made be discussed in most elementary courses. *Indeed, in the second derivations one only needs to state that the region J is one of those studied in the elementary calculus.*

$\diamondsuit$ Let compact and Jordan-measurable $J \subset R \subset \mathbb{R}^3$ and S be an inner rectangle as defined above. Let $\rho\colon J \to \mathbb{R}^3$ be a continuous density function. If $\rho(x,y,z)$ is constant over a subrectangle S of J, then from the definition of ρ the mass $m(S) = \rho\, v(S)$, where $v(S)$ is the volume of S. Suppose that ρ is not constant. Then there exists $(x_m, y_m, z_m) \in S$ and $(x_M, y_M, z_M) \in S$ such that $\rho_m = \rho(x_m, y_m, z_m) \leq \rho(x,y,z) \leq \rho(x_M, y_M, z_M) = \rho_M$ for each $(x,y,z) \in S$. Hence, if the mass of $S = m(S)$, then observation and the above property for constant density indicates that, at least, from the macroscopic viewpoint

$$\rho_m\, v(S) \leq m(S) \leq \rho_M\ v(S). \tag{1}$$

The Intermediate Value Theorem for connected subsets of R yields that there is some $(x_1, y_1, z_1) \in S$ such that $m(S) = \rho(x_1, y_1, z_1)\, v(S)$. If we have more than one such inner rectangle, say $\{S_1, \ldots, S_n\}$, than observation indicates that the total mass, $m(A)$, of the configuration $A = S_1 \cup \cdots \cup S_n$ is the finite sum

$$m(A) = \sum_{j=1}^{n} m(S_j) = \sum_{j=1}^{n} \rho(x_j, y_j, z_j)\, v(S_j). \tag{2}$$

By *-transfer, these results hold for infinitesimal subrectangles of J and hyperfinite summation. Now let Q be any fine partition of ${}^*R$. Then $\mathrm{inn}(Q)$ is an internal hyperfinite set of infinitesimal subrectangles. Hence,

$${}^*m(\cup\{S_j | S_j \in \mathrm{inn}(Q)\}) = \sum_{j=1}^{\Gamma} {}^*\rho(x_j, y_j, z_j)\ {}^*v(S_j). \tag{3}$$

Even though each point $\vec{v}_j = (x_j, y_j, z_j) \in S_j$, this does not imply that the intermediate partition $\{\vec{v}_j \bigm| 1 \leq j \leq \Gamma\}$ is internal. However, letting $\vec{c}_j$ denote the corner of S_j nearest or equal to $(0,0,0)$ then $\{\vec{c}_j \bigm| 1 \leq j \leq \Gamma\}$ is an internal intermediate partition. Uniform continuity of ρ on J yields that for each j there exists some $\epsilon_j \in \mu(0)$ such that ${}^*\rho(x_j, y_j, z_j) = {}^*\rho(\vec{c}_j) + \epsilon_j$. Hence,

$${}^*m(\cup\{S_j | S_j \in \mathrm{inn}(Q)\}) = \sum_{j=1}^{\Gamma} ({}^*\rho(\vec{c}_j) + \epsilon_j)\ {}^*v(S_j). \tag{4}$$

The elemental derivation process yields

$$ {}^*m(\cup\{S_j|S_j \in \mathrm{inn}(Q)\}) \approx \sum_{j=1}^{\Gamma}(\,{}^*\rho(\vec{c}_j)\ {}^*v(S_j). \tag{5} $$

Since ρ is integrable over J then taking the standard part and applying Theorem 7.2.2 of the Infinitesimal Modeling manual one obtains

$$ \mathtt{st}(\,{}^*m(\cup\{S_j|S_j \in \mathrm{inn}(Q)\})) = \int_J \rho(x,y,z)\,dX. \tag{6} $$

As mentioned above, one final step is required. It must be assumed that the concept of the mass of an object has the same Jordan-like quality as does the concept of the volume. Making this last assumption implies that $m(J) = \int_J \rho(x,y,z)\,dX.\diamondsuit$

A physical interpretation of the Jordan-like quality of the mass is also possible. For every fine partition Q, the NSP-world effects of the mass of a simple internal rectangular configuration $\cup\{S_j \bigm| S_j \in \mathrm{inn}(Q)\}$ is indistinguishable from the effects of the mass of J at the first (order) level, if no comparison is to be made with other such measures. When such a comparison is made, as pointed out in the Infinitesimal Modeling manual, Section 8.3, the effects of ${}^*m(\cup\{S \bigm| S \in \mathrm{inn}(Q)\})$ and $m(J)$ would be infinitely close of order 3 and indistinguishable at that level. Of course, all that is being derived for the case $\mathbb{R}^3$ holds for ${}^*m(\cup\{S \bigm| S \in \mathrm{inn}(Q)\})$ and any $\mathbb{R}^n,\ n \geq 1$.

In all of the previous derivations, equations of form (1) are the essential modeling requirements. The effect of the elemental derivation process is to eliminate the additional additivity property for such functionals m. However, if such additivity is assumed, then the integral expression follows immediately from the self-evident theorems that appear in the Infinitesimal Modeling manual.

{*Second derivation.* Now that we have the essential procedures needed to modify a rigorous infinitesimally styled derivation and obtained a quasi-classical one, it is not necessary to present an entire second derivation for equation (6). It is useful to conjoin the following to our list of alterations. The set J can simply be described as one of the regions studied in the elementary calculus and for which an integral expression for volume is obtained. For the necessity of expressing $\rho(x_j,y_j,z_j)$ in equation (3) as $\rho(x_j,y_j,z_j) = \rho(\vec{c}_j) + \epsilon_j$, without invoking the concept of the internal intermediate partition, one can simply argue that it is necessary to evaluate ρ at a known partition generated point rather than at a less explicitly known $(x_j,y_j,z_j) \in S$. The requirement that m be Jordan-like can be described as a common feature that m must share with the volume of the region J. The feature being, that for any infinitesimal partition the configuration composed of all of the infinitely small subrectangles contained within J must have its volume infinitely close the volume of J. This same feature holds for the mass and other such measures since if $\rho \equiv 1$ on J, then $v(J) = m(J)$. I leave to the reader the actual construction of the second derivation.}

3.2 Moments and Center of Mass

It is often difficult to decide whether it is more significant to derive a specific mathematical model completely from fundamental physical observations and the defining properties of the mathematical structure or to replace portions of the derivation with well-known theorems gleaned from the abstract structure itself. It appears that within modern theoretical physics, abstract mathematical results are introduced as soon as practical even though they may be couched in a quasi-physical language. This procedure follows the routine assumption that the rigorous logic displayed when a proposition is proved abstractly is equivalent to the logic required for a much longer and more complex derivation that utilizes but the fundamental correspondence between the defining properties of the structure and the physical terms that describe the physical scenario. The following theorem is useful in order to illustrate the economy achieved by such an early introduction of well-known mathematical results.

Theorem 3.2.1. A Weighted Mean Value Theorem. *Let $J \subset E \subset R \subset \mathbb{R}^n$, J be Jordan-measurable, and E be compact and connected. Let continuous $f\colon E \to \mathbb{R}$, integrable $g\colon J \to \mathbb{R}$ and $g(\vec{v}) \geq 0$ for each $\vec{v} \in J$. Then there exists some $\vec{v}_0 \in E$ such that $\int_J f(\vec{v})g(\vec{v})\,dX = f(\vec{v}_0)\int_J g(\vec{v})\,dX$.*

Even though the elemental derivation process is being stressed throughout this physics manual, it is worthwhile to once again mentioned that the Self-Evident theorems of the Infinitesimal Modeling manual are always available. The Method of Constants, as well as the Maximum-Minimum Method, is actually exemplified within the elemental process derivations. Immediately following equation (1) of this section is the fact that there does exit some $(x_1, y_1, z_1) \in S$ such that $m(S) = \rho(x_1, y_1, z_1)v(S)$. This is the explicit requirement for application of Proposition 6.9 in Appendix 6 assuming the requisite additivity properties for the measure m. If the measure generating function is composed of the product of two or more nonconstant functions, then the Extended Self-Evident Method of Constants Proposition 6.10 may be appropriate.

$\diamondsuit$ Notwithstanding our discussion in the above paragraph, let's consider the a moment generating function $M_{(\cdot)}$ defined as follows: Let nonnegative continuous $\rho(x, y, z)\colon J \to \mathbb{R}$, where $J \subset R \subset \mathbb{R}^3$ is compact and Jordan-measurable. A moment function defined on a partition subrectangle $S \subset J$ is $M_{yz}(S) = x\,\rho(x, y, z)\,v(S)$, where $x = f(x, y, z)$ is continuous on $\mathbb{R}^3$. In the same manner, define $M_{xz}(S) = y\,\rho(x, y, z)\,v(S)$, $M_{xy}(S) = z\,\rho(x, y, z)\,v(S)$. Although it may appear to be sufficient to consider the point (x, y, z) as an arbitrary member of S, certain special selections are necessary. For example, if ρ is constant, then (x, y, z) might be selected as the center of S. It is well-known that if we consider a small enough homogeneous rectangular solid S, then the gravitational field of the Earth in relation to S is effectively a parallel vector field and the acceleration of gravity is a constant g. The center of the rectangle is the point of rotational stability within such a field — the so-called center of gravity. This is demonstrated in the customary manner by considering the expressions $g\,M_{xy}(S)$, $g\,M_{yz}(S)$, $g\,M_{xz}(S)$. Thus from the Newtonian gravitational point of view, the moment function can be viewed as a measure of the rotational effect with respect to the coordinate planes within such a gravitational field. Notice that the necessary observations that lead to this conclusion are relative to the actual small size of the objects.

For the general case of continuous ρ on S, there exist $x_m\rho(x_m, y_m, z_m)$ and $x_M\rho(x_M, y_M, z_M)$, where (x_m, y_m, z_m), $(x_M, y_M, z_M) \in S$ such that

$$x_m\rho(x_m, y_m, z_m)v(S) \leq x\,\rho(x, y, z)v(S) \leq x_M\rho(x_M, y_M, z_M)v(S). \tag{1}$$

The usual assumption is now imposed upon our problem. Suppose that the actual moment effect $M_{yz}(S)$ lies somewhere between these two extremes. Thus there would exist some $(x_1', y_1', z_1') \in S$ such that

$$M_{yz}(S) = x_1'\,\rho(x_1', y_1', z_1')\,v(S). \tag{2}$$

Equation (2) is now extended to a finite system $\{S_j \mid 1 \leq j \leq n\}$ of partition inner subrectangles using the apparent experiential result that the total moment effect of the system is the sum of the individual effects. Hence,

$$M_{yz}(\cup S_j) = \sum_{j=1}^{n} x'_j\, \rho(x'_j, y'_j, z'_j)\, v(S_j). \tag{3}$$

By *-transfer, we assume that within the NSP-world the above behavior holds for infinitesimal rectangles. Since this is a modeling technique it is not necessary to assume that such infinitesimal rectangles exist in some type of objective reality. *However, it is possible to describe such behavior within a substratum NSP-world by considering the simplistic behavior of a hyperfinite set of NSP-world infinitesimal rectangles as a superstructure of objects that controls the behavior of a corresponding system of "small" natural world rectangular objects.* As in the least section, assuming that Q is a fine partition of $^*R$ this leads to the conclusion that

$$^*M_{yz}(\cup\{S_j | S_j \in \mathrm{inn}(Q)\}) = \sum_{j=1}^{\Gamma} x'_j\ {}^*\rho(x'_j, y'_j, z'_j)\ {}^*v(S_j). \tag{4}$$

Letting (x_j, y_j, z_j) denote the corner of S_j nearest or equal to $(0,0,0)$ then $\{(x_j, y_j, z_j) \bigm| 1 \leq j \leq \Gamma\}$ is an internal intermediate partition. Uniform continuity of $x\rho$ on J yields that for each j there exists some $\epsilon_j \in \mu(0)$ such that $x'_j\ {}^*\rho(x'_j, y'_j, z'_j) = x_j\ {}^*\rho(x_j, y_j, z_j) + \epsilon_j$. Applying the elemental derivation process one obtains

$$\mathtt{st}(\,^*M_{yz}(\cup\{S_j | S_j \in \mathrm{inn}(Q)\})) = \int_J x\, \rho(x, y, z)\, dX. \tag{5}$$

The additional requirement that the measure of the moment is Jordan-like yields $M_{yz}(J) = \int_J x\, \rho(x, y, z)\, dX$.

Now $R \subset \mathbb{R}^3$ is compact and connected and $f(x, y, z,) = x$ is continuous on R. Thus, by the Weighted Mean Value Theorem there exists some real $\overline{x}$ such that

$$M_{yz}(J) = \int_J x\, \rho(x, y, z)\, dX = \overline{x} \int_J \rho(x, y, z)\, dX = \overline{x}\, m(J). \tag{6}$$

Repeating the above argument there exist real $\overline{y}$ and $\overline{z}$ such $M_{xz}(J) = \int_J y\, \rho(x, y, z)\, dX = \overline{y} \int_J \rho(x, y, z)\, dX = \overline{y}\, m(J)$ and $M_{xy}(J) = \int_J z\, \rho(x, y, z)\, dX = \overline{z} \int_J \rho(x, y, z)\, dX = \overline{z}\, m(J)$. Consequently, as far as the moment effects are concerned the object J can be consider as represented by the single point $(\overline{x}, \overline{y}, \overline{z})$ with the mass number $m(J)$ attached to it.$\diamondsuit$

{*Second derivation.* From this point on in this Elementary Physics manual, the second quasi-classical derivation will not be given unless it is substantially different from our previous examples.}

Notice that the above derivation did not start with the concept of the point masses and then derive the integral expression for the center of mass. Rather, we derived the concept by means of infinitesimal analysis. Additionally, the statement that the center of mass is equivalent to the center of gravity appears in the above discussion to depend upon the parallel gravitational field concept. In the next section, we show that the idea of the less substantiated point masses, *if viewed from the NSP-world,* does lead to the same center of mass conclusion.

3.3 Point Masses

In `Tipler [1982]`, the concept of the point mass is used to develop the center of gravity and center of mass for such objects, assuming that this technique has been justified. On page 229 of volume 1, Tipler states: *"If the center-of-mass coordinates of a continuous body are to be calculated,* *the sum $\sum m_i x_i$ must be replaced by the integral $\int x\,dm$, where dm is an element of mass."* In the text by Young, Riley, McConnell, Rogge [1974, p. 281], when moments of inertia are discussed, once again the student is instructed that such a measure is given by an integral over dm. No further explanation is given as to why this particular technique is justified. This vague modeling technique can be justified within the NSP-world once equations such as (6) of Section 3.2 have been derived.

$\diamondsuit$ Assume the hypotheses used to derive (6) of Section 3.2 and let Q once again be a fine partition of ${}^*R$. Let $S_j \in \text{inn}(Q)$ Then there exists some $(x''_j, y''_j, z''_j) \in S_j$ such that ${}^*m(S_j) = {}^*\rho(x''_j, y''_j, z''_j)\ {}^*v(S_j)$. Now ${}^*\rho(x''_j, y''_j, z''_j) \approx {}^*\rho(x'_j, y'_j, z'_j)$, where (x'_j, y'_j, z'_j) is as described in the derivation of Section 3.2. Hence, $x'_j\,{}^*\rho(x'_j, y'_j, z'_j) \approx x'_j\,{}^*\rho(x''_j, y''_j, z''_j)$, for x'_j is limited. The elemental derivation process yields

$$M_{yz}(J) = \int_J x\,\rho(x,y,z)\,dX = \mathtt{st}\left(\sum_{j=1}^{\Gamma} x'_j\,{}^*\rho(x''_j, y''_j, z''_j)\ {}^*v(S_j)\right). \tag{1}$$

Consequently, letting $dm_j = {}^*m(S_j) = {}^*\rho(x''_j, y''_j, z''_j)\ {}^*v(S_j)$ denote that mass of the infinitesimal subrectangle (not a point!) and repeating the argument for the other two moments one can describe the moment effects within the standard world as follows: The effect is indistinguishable from the effect of a hyperfinite sum of mass numbers attached to the points (x'_j, y'_j, z'_j). Thus the points (x'_j, y'_j, z'_j) within the NSP-world can be viewed as point masses. Obviously, the point (x'_j, y'_j, z'_j) is not unique since it may be replaced by any $(x,y,z) \in S_j$.$\diamondsuit$

The reader might be inclined to attempt to *-transfer the above derivation to the standard world and arrive at the conclusion that there exists a set of point masses that would yield the moment effects expressed by the integrals. This would be an error, however, since the standard part operator and as well as $\approx$ are external concepts. The technique of *-transfer, at this stage, only allows the hyperfinite sum of point masses to be transferred into a statement about finitely many point masses the sum of the moments of which would approximate the moment effect within an given positive ϵ for all the partitions of R with mesh less than some positive δ. What this implies is that rather than accepting an ad hoc modeling technique that utilizes unrealistic standard world point masses to derive the integral expressions for the moment effects, it may be more conducive to student comprehension to employ the NSP-world point masses since their use can be more rigorously justified. *However, in certain cases once infinitesimal analysis has established equation (6) of Section 3.2, and the like, then* <u>*standard means*</u> *can be applied to investigate an effective center of mass for a finite collection of objects.* This we do next.

$\diamondsuit$ Suppose that $\{J_1, \ldots, J_p\}$ is a nonempty *finite* set of of pairwise disjoint, compact, and Jordan-measurable subsets of $R \subset \mathbb{R}^3$. Further, let nonnegative continuous $\rho_i\colon J_i \to \mathbb{R}$, for each i such that $1 \le i \le p$. The function $f(x,y,z) = x$ is obviously continuous on compact, connected R. The piecewise well-defined function $h(x,y,z) = \rho_i(x,y,z);\ (x,y,z) \in J_i,\ 1 \le i \le p$ is continuous on $J = J_1 \cup \cdots \cup J_p$. [Since $\mathbb{R}^3$ is a normal topological space it follows that if $\vec{v} \in J_i$, then $\mu(\vec{v}) \cap J_j = \emptyset,\ i \ne j$.] Now the set J is compact; hence, closed and bounded. Consequently, h is a nonnegative bounded function defined on J. The function $f(x,y,z) = x$ is continuous on compact, connected R and integrable on J. Consider the moment effect generating function $m_{yz} = x\,h(x,y,z)$. Then from the Weighted Mean Value Theorem there exists $\overline{x}_j$ such that the moment effect

$$M_{yz}(J) = \int_J m_{yz}\,dX = \overline{x}_j\,m(J). \tag{2}$$

However,

$$\int_J m_{yz}\, dX = \sum_{i=1}^{p} \int_{J_i} x\rho_i(x,y,z)\, dX = \sum_{i=1}^{p} \overline{x}_i\, m(J_i). \tag{3}$$

Thus

$$M_{yz}(J) = \sum_{i=1}^{p} \overline{x}_i\, m(J_i). \tag{4}$$

Repeating the above derivation yields similar equations as (4) for the other two moments. However, I repeat, once again, that this approach is only relative to a nonempty finite set of disjoint, compact, and Jordan-measurable subsets of R. $\diamondsuit$

3.4 Standard Rules and the Elemental Derivation Process

Although the elemental derivation process is very appealing to the intuition, what happens when this process is viewed as a mathematically stated theorem? If you were to analyze the standard hypotheses needed to model this process, then what would be obtained is the rule IR5 as it appears in the Infinitesimal Modeling manual. This rule coupled with the hypotheses stated in a theorem such as Proposition 6.7 in the Infinitesimal Modeling manual leads to a formal theorem that can be established not by an infinitesimal sum theorem but be the elemental derivation method. We develop such a theorem next — a theorem that allows us to eliminate the actual elemental derivation process. The necessary notation for what follows is defined in this manual.

We will not state what comes next as a formal theorem but state it somewhat informally. We point out that it is but a restatement in a slightly expanded form of Proposition 6.7 in the Infinitesimal Modeling manual. Let $\mathcal{P}$ be any set of partitions of the rectangle $R \subset \mathbb{R}^n$ and compact Jordan-measurable $J \subset R$. Suppose that ${}^*\mathcal{P}$ contains a fine partition. Of course, if $\mathcal{P}$ is the set of all simple partitions of R, then such a fine partition exists. Next let B be a real valued function(al) defined on $\{I_\mathcal{P}, J\}$ and, at least, additive on $\{I_\mathcal{P}, J\}$. Let continuous $f\colon J \to \mathbb{R}$. Suppose that for any $P \in \mathcal{P}$ and any $S \in \mathrm{inn}(P)$, it follows that $(f_m)v(S) \leq B(S) \leq (f_M)v(S)$, where (f_m) [resp. (f_M)] is the minimal [resp. maximal] value of f on S and $v(S)$ is the Jordan-measure of S (i.e. its simple volume). Then if B has the ordinary Jordan-like property, it follows (from the elemental derivation process) that $B(J) = \int_J f(\vec{x})\, dX$.

What the last paragraph signifies is that in all cases where elementary physical measures are concerned one needs only argue for the acceptance of the stated hypotheses. Once such hypotheses are accepted as reasonable, then the conclusion follows from both the infinite sum theorem or the elemental derivation process.

The assumption that B has the ordinary Jordan-like property is not difficult to accept. A standard criterion appears in the Infinitesimal Modeling manual and that property is apparently necessary in order for B to be obtained by an integral. This comes from the fact that the integral itself when viewed as a functional satisfies this property. It intuitively signifies that an approximation for the value of $B(J)$ that is better that any machine error can be obtained by considering the value $B(C)$, where $C \subset J$ is a configuration composed of subrectangles taken from a partition with "small enough" mesh. Now compare this with the requirements of IR3 (1) in the Infinitesimal Modeling manual and Theorem 6.2.4, where the ordinary Jordan-like property is not assumed. In this case, the maximal–minimal assumption is weakened for boundary rectangles. This weakening is relative to the value of the functional as extended to the boundary rectangles. In an elementary exposition, it may be more reasonable to "build" configurations such as C and accept the ordinary Jordan-like property for such configurations, then to alter the intuitive acceptance of such maximal-minimal statements as (1) on page 22 of this manual.

Thus far, we have needed to include the strong requirements that the function f be continuous on J and that J be, at least, compact and Jordan-measurable. Can either or both of these requirements be relaxed and an acceptable derivation method developed for integral models? An answer to this question will depend upon what one considers as "acceptable" and the areas of application. We will attempt to answer this question in later sections of this manual.

To be continued by properly trained members of the physics community.

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Additional Special Symbols

(Alphabetically listed by first symbol letter.)

Symbol	Name, if any	Page no.
$\sim_1$	Infinitely Close of Order One	170
$\sim_2$	Infinitely Close of Order two	172
$\vec{I}$	Impulse Vector	177
$m(S)$	Mass of S	182
$M_{(\cdot)}$	Moment Function	184
dm	Mass Of Infinitesimal $S =$ Element Of Mass	186

NONSTANDARD ANALYSIS AND GENERALIZED FUNCTIONS

Robert A. Herrmann

1. Additional Modeling Concepts.

In what follows, the basic notation and definitions are as they appear in [3]. However, except as mentioned in the appendix of [3], a significant aspect of nonstandard analysis has not been developed fully. The superstructure constructed in [3] is a model for Γ where Γ is the set of all sentences that hold in the structure. Since it is constructed from $\mathbb{R}$, it is called a model (for real analysis), for apparently every true statement from analysis holds true in the superstructure. The elementary nonstandard structure ${}^*\mathcal{M} = ({}^*\mathcal{H}, \in, =)$ is associated with the standard model $\mathcal{M} = (\mathcal{H}, \in, =)$ in a slightly special sense. Due to the applications in [3], it was not necessary to discuss ${}^*\mathcal{M}$ relative to its special properties. This is no longer the case.

The structure ${}^*\mathcal{M}$ is constructed from a (bounded) ultrapower based upon the structure $\mathcal{M}$. [1, p. 15–19], [2], [5, p. 83–88] It is assumed that there are constants that denote every member of $\mathcal{H}$ where we do not differentiate between a constant and the object it names. Suppose that J is the index set and that $\mathcal{U}$ is an appropriate ultrafilter on J. Let infinite $A \in \mathcal{H}$ and A contains no individuals in $\mathbb{R}$. Let $f \in \mathcal{H}^J$ and $f(j) = A$, for each $j \in J$. This is the constant map, constant in two ways both as a mathematical entity and relative to the value being denoted by a constant in our language. Some authors define an injection k, which is denoted by e in [5] and i in [1], such that $k(A) = [A]$, where $[A]$ is the $\mathcal{U}$-equivalence class in the ultrapower that contains the constant map f. To obtain an isomorphic copy of $\mathcal{M}$, we could follow the usual Mostowski collapsing process as described in [5, p. 84–85] that gives the * mapping and let this map be restricted to the domain of all the constant sequence $\mathcal{U}$-equivalence classes in the ultrapower. This leads to an isomorphic copy of $\mathcal{M}$. [5, p 85.] But, care must be taken relative to the interpretation of the * map. It must always be remembered the ${}^*\mathcal{M}$ is a model of the bounded expressions that hold in $\mathcal{M}$ although sometimes the bounding set is not expressly stated in an expression it must be understood that the quantifiers are restricted to specific elements in some X_n for $\mathcal{M}$ and for the corresponding ${}^*X_n$ for ${}^*\mathcal{H}$. The members of ${}^*X_n$ are "internal" entities.

Consider the first-order statement $a \in b$ about objects in $\mathcal{H}$. Then the statement ${}^*a \in {}^*b$ holds in ${}^*\mathcal{M}$ and is relative to the embedding of $\mathcal{M}$ into ${}^*\mathcal{M}$. Now consider a set $A \in \mathcal{H}$ and the bounded statement $S = \forall x((x \in A) \wedge (x \in b))$, which is in the required form, and let $S' = \{x \mid (x \in A) \wedge (x \in b)\}$. Then S holds in $\mathcal{M}$ if and only if ${}^*S = \forall x((x \in {}^*A) \wedge (x \in {}^*b))$ holds in ${}^*\mathcal{M}$. This indicates that the quantifiers are restricted to members of ${}^*A$. Since A is a set, $A \in X_n$ for some $n \geq 1$. Then from Proposition 1 (iv) [3], it follows that each specific "x" that satisfies ${}^*S'$ is a member of ${}^*X_0$ or ${}^*X_n$, where $n \geq 1$. Thus each such "x" is a member of ${}^*\mathcal{H}$.

On the other hand, if S holds in $\mathcal{M}$, then S also holds in the isomorphic copy of $\mathcal{M}$, where the quantifier is restricted to constant sequence $\mathcal{U}$-equivalence classes. Hence the only members in the isomorphic copy of the set S' are the ${}^*a$ such that $({}^*a \in {}^*A) \wedge ({}^*a \in {}^*b)$. But is this isomorphic copy of S' a member of ${}^*\mathcal{H}$? One of the ideas behind the concept of a nonstandard structure is that if S' is infinite, then its isomorphic copy is not a member of ${}^*\mathcal{H}$ although it is a subset of ${}^*A$ and ${}^*A \in {}^*\mathcal{H}$. In this case, the set ${}^*S'$ contains entities that are not produced by the * map. In order to discuss sets such as the isomorphic copy of S', a second superstructure $(\mathcal{Y}, \in, =)$ is generated with ground set $Y_0 = {}^*X_0$. It is within this superstructure that we can specifically construct the sets that restrict the quantifiers when bounded statements are interpreted for the isomorphic copy of $\mathcal{M}$.

Definition 1.1 (Sigma Operator, Standard Copy) If a set $A \in \mathcal{H}$, then ${}^\sigma A = \{ {}^*a \mid a \in A\}$, where the ${}^*a$ in this form denotes the constant sequence $\mathcal{U}$-equivalence class [8, p. 44-45].

Let ${}^\sigma\mathcal{H} = \{{}^\sigma A \mid A \in \mathcal{H}\}$. There is a certain confusion of symbols that one tries to avoid. When the isomorphic embedding is being considered, it is understood that ${}^*a$ means something different when viewed as a *set* of objects. The symbol ${}^*a$ is used only as a name for the equivalence class in the ultrapower that contains the constant sequence for the isomorphic embedding. But, relative to the structure ${}^*\mathcal{M}$, and as a nonempty set, ${}^*a$ contains the objects in ${}^\sigma a$. Thus relative to the sets in $(\mathcal{Y}, \in=)$, the isomorphic copy is determined by the ${}^\sigma$ and * operators. It is this fact that has lead to a minimizing of the use of the ${}^\sigma$ notation. Thus it is customary to do our real analysis in $\mathcal{M}$ rather then in ${}^\sigma\mathcal{M} = ({}^\sigma\mathcal{H}, \in, =)$ knowing that, when *comparisons* are to be made, we can apply the isomorphism to obtain the actual objects in ${}^\sigma\mathcal{H}$ that are used for modeling purposes. I mention that significant fact is that if a set $A \in \mathcal{H}$ is finite, then ${}^*A = {}^\sigma A$. One might say that the map ${}^\sigma$ preserves the hierarchy of finite sets. (Some authors find no need to consider ${}^\sigma\mathcal{M}$ [5].)

From the viewpoint of abstract model theory, what are the real numbers? The real numbers is considered to be ANY structure isomorphic to the "standard" structure. The standard structure is considered to be a nonempty set $\mathbb{R}$ with the appropriate operators and unique elements defined and such that the operators, unique elements and subsets of $\mathbb{R}$ satisfy a set of axioms. Under the above isomorphism *, Theorem 3.1.1 in [3] implies that the isomorphic copy of the real numbers can be considered as THE real numbers and real analysis takes place in ${}^\sigma\mathcal{H}$. Thus, as has become customary, we let ${}^\sigma\mathbb{R} = \mathbb{R}$. Moreover, to more fully express this identification of the real numbers, consider how this isomorphism deals with the operator $\subset$. From the definition of the operator $\subset$, it follows that if $A \in \mathbb{R}$, then ${}^\sigma A = \emptyset$. Further, given two $A, B \in \mathcal{H}$. Then $A \subset B$ if and only if ${}^\sigma A \subset {}^\sigma B$. Hence, if $A \subset \mathbb{R}$, then $A = {}^\sigma A \subset {}^*A$. Basic operators such as $+,\ \cdot,\ <$, under the isomorphism, become the operators ${}^\sigma+,\ {}^\sigma\cdot,\ {}^\sigma <$ which then become THE operators $+,\ \cdot,\ <$ for the field $\mathbb{R}$. Although the ${}^\sigma$ notation could continue to be used on sets at any point when one is in doubt, it has become customary to remove this notation in some cases like ${}^\sigma\mathbb{R} = \mathbb{R}$. Often relations such as ${}^*+,\ {}^*\cdot,\ {}^*<$ for the ordered field ${}^*\mathbb{R}$ are considered as the actual extensions of the relations $+,\ \cdot,\ <$.

Relative to notation, what this means is that many of the constants in $C(\mathcal{H})$ that denote the members of $\mathcal{H}$ will now be used to denote many members of ${}^\sigma\mathcal{H}$. New symbols ${}^\sigma A$ are used to denote other members of ${}^\sigma\mathcal{H}$. Then we have objects in ${}^*\mathcal{H}$ that have names in the symbol set $C({}^*\mathcal{H})$ and represent the internal extended standard objects. These are denoted by use of the * notation. All other members of ${}^*\mathcal{H}$, other than the internal members, are denoted by distinctly different symbols in $C({}^*\mathcal{H})$. Since there are only so many symbols that can be used, we must state that this non-"started" symbol represents an "internal" object. Each object in $(\mathcal{Y}, \in, =)$ has a symbol name. All such objects that are not denoted by any previously defined notation must be explicitly defined as "external" objects. In this regard, if infinite $A \in \mathcal{H}$, then for the actual structure ${}^*\mathcal{M}$ soon to be constructed, the object (denoted by) ${}^\sigma A$ is external.

Relative to mathematical modeling, if we are modeling physical entities with names taken from a specific discipline dictionary, then it is immaterial which real analysis structure is used for the modeling correspondence. Thus, the isomorphic ${}^\sigma\mathcal{M}$ is chosen as the appropriate structure. Hence, if the Euclidean n-space function $\mathbf{v}(t)$ corresponds to a natural physical world (i.e. N-world) vector, then the function ${}^*\mathbf{v}(t)$ corresponds to a nonstandard physical world (i.e. NSP-world) vector. Since

$\mathbf{v} \subset {}^*\mathbf{v}$, and $\mathbf{v}, {}^*\mathbf{v} \in (\mathcal{Y}, \in, =)$, the N-world physical vector can be assumed to be a restriction of the NSP-world vector to the N-world. Although more can be said about the effects of such a restriction relative to direct and indirect observation, it is not necessary, at this point, to delve more deeply into such concepts.

There has also arisen a certain terminology. Suppose that f is a continuous map from $\mathbb{R}$ into $\mathbb{R}$ (i.e. $f\colon \mathbb{R} \to \mathbb{R}$). Then one can write that ${}^*\!f$ is a *-continuous map from ${}^*\mathbb{R}$ to ${}^*\mathbb{R}$. The term "*-continuous" is often replaced in scientific discourse by the term "hypercontinuous." On the other hand, some authors leave the term in this form and when communicating orally say "hypercontinuous" or "star-continuous."

Definition 1.2 (Concurrent Relation) A (bounded) binary relation Φ in $\mathcal{H}$ and, hence, in ${}^\sigma\mathcal{H}$, is *concurrent* if the following holds. For each finite $\neq \emptyset$ set $A = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subset \Phi$ there exists in the range, $R(\Phi)$, of Φ some b such that $\{(a_1, b), \ldots, (a_n, b)\} \subset \Phi$.

Now as to the actual construction of the nonstandard structure ${}^*\mathcal{M}$, a special ultrafilter is selected which has the following *enlarging* property.

Definition 1.3 (Enlargement) The structure ${}^*\mathcal{M}$ is an *enlargement* if for every concurrent relation Φ with the domain $D(\Phi)$ there exists an internal $b \in {}^*(R(\Phi)) = R({}^*\Phi)$ such that for each ${}^*a \in {}^\sigma D(\Phi)$, $({}^*a, b) \in {}^*\Phi$.

In all that follows, it is assumed that ${}^*\mathcal{M}$ is, at the least, an enlargement. Also note that all members of ${}^*(R(\Phi))$ are internal.

Theorem 1.1. *Let infinite $A \in \mathcal{H}$. Then ${}^\sigma A \subset {}^*A$ and ${}^\sigma A \neq {}^*A$.* Proof. Suppose that infinite $A \in \mathcal{H}$. Consider the relation $\Phi = \{(x, y) \mid (x \in A) \wedge (y \in B) \wedge (x \neq y)\}$. Suppose that $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subset \Phi$. However, since A is infinite, there exists some $b \in A$ such that $b \neq a_i,\ i = 1, \ldots, n$. Hence, $\{(a_1, b), \ldots, (a_n, b)\} \subset \Phi$. Thus Φ is a concurrent relation. Consequently, there is an internal $b \in R({}^*\Phi)$ such that $({}^*a, b) \in {}^*\Phi$ and $b \neq {}^*a$. This completes the proof.

Often just one identified concurrent relation can determine a major portion of an entire nonstandard theory. A few more examples indicate this fact. First, consider the extension of the absolute value function to ${}^*\mathbb{R}$. By definition, for any $r \in \mathbb{R}$ if $r \geq 0$, then $|r| = r$ and for any $r \in \mathbb{R}$ if $r < 0$, then $|r| = -r$. Stated formally, we have that $S = \forall x(((x \in \mathbb{R}) \wedge (r \geq 0)) \to (|r| = r)) \wedge ((x \in \mathbb{R}) \wedge (r < 0)) \to (|r| = -r)))$. This statement holds in $\mathcal{M}$. Thus its *-transform holds in ${}^*\mathcal{M}$. The *-transform is ${}^*S = \forall x(((x \in {}^*\mathbb{R}) \wedge (r \geq 0)) \to ({}^*|r|) = r) \wedge ((x \in {}^*\mathbb{R}) \wedge (r < 0)) \to ({}^*|r|) = -r))$, where $|\cdot|$ is viewed as a unary operator. Thus the operator ${}^*|\cdot|$ is but the absolute value operator as it is defined for the totally ordered field ${}^*\mathbb{R}$. Hence we can drop the * notation from ${}^*|\cdot|$.

Theorem 1.2. *There exists in ${}^*\mathbb{R}$, a nonzero infinitesimal.*

Proof. Consider the relation $\Phi = \{(n, m) \mid (0 < (1/m) < (1/n)) \wedge (n \in \mathbb{N}) \wedge (m \in \mathbb{N})\}$. Suppose that $\{(n_1, m_1), \ldots, (n_j, m_j)\} \subset \Phi$. Let $M = \max\{m_1, \cdots, m_j\}$. Then $\{(n_1, M{+}1), \ldots, (n_j, M{+}1)\} \subset \Phi$ implies that Φ is concurrent. Hence, there exists some $\Lambda \in {}^*\mathbb{N}$ such that $0 < 1/\Lambda < 1/n$ for all $n \in {}^\sigma\mathbb{N} = \mathbb{N}$. Now consider any positive $r \in \mathbb{R}$. Then there exists some $n \in \mathbb{N}$ such that $0 < 1/n < r$.

Hence, $|1/\Lambda| < r$. But since r is an arbitrary positive real number, this last results hold for all $r \in \mathbb{R}^+$. This completes the proof.

An examination of chapter 2 in [3] shows that the properties of the set of all infinitesimals $\mu(0)$ are determined from Theorem 1.2. One of the most significant portions of the nonstandard theory of analysis is relative to the set of all "hyperfinite" sets. These are all of the internal sets $A \in {}^*(F(\mathcal{H}) = \cup\{{}^*(F(X_n)) \mid X_n \in \mathcal{H}\}$. Also note that if $A \in X_n,\ (n > 0)$, then $F(A) \in X_{n+1}$. Hence, ${}^*(F(A)) \in {}^*X_{n+1}$. Viewed as a mapping, we have that ${}^*F$ is defined on all internal sets in ${}^*\mathcal{H}$ and if $B \in \mathcal{H}$, then ${}^*(F(B)) = {}^*F({}^*B)$. As shown in [3] Theorem 4.3.2, the above definition for the hyperfinite sets is equivalent to the the internal bijection definition. The hyperfinite sets satisfy in ${}^*\mathcal{M}$ all of the first-order properties associated with finite sets. But from the exterior viewpoint of nonstandard analysis, such sets are far from being finite.

To get the full strength of nonstandard analysis as it relates to generalized functions, we need to concept of the κ-saturated enlargement, where κ is a infinite cardinal number.

Definition 1.4 (κ-Saturated) The structure ${}^*\mathcal{M}$ is a *κ-saturated* if given any internal (bounded) binary relation Φ, with the internal domain $D(\Phi)$, that is concurrent on $A \subset D(\Phi)$, where cardinality of $A < \kappa$, then there exists an internal $b \in R(\Phi)$, the internal range of Φ, such that for each $a \in A,\ (a, b) \in \Phi$.

Throughout the remaining portions of this paper, we assume that ${}^*\mathcal{M}$ is a κ-saturated, where κ is any (regular) cardinal number greater than the cardinality of $\mathcal{H}$. This would also imply that ${}^*\mathcal{M}$ is an enlargement. It can be shown by means of the ultralimit process, such (bounded) κ-saturated enlargements exist.

Theorem 1.3. *Consider internal B and any $A \subset B$ such that cardinality of $A < \kappa$. Then there exists a hyperfinite set Ω such that $A \subset \Omega \subset B$.*

Proof. From the construction of $\mathcal{H}$ we know that there is some $X_n,\ n \geq 1$ and $B \in {}^*X_n$, ${}^*F(B) \in {}^*X_n$, if $q \in B,\ \{q\} \in X_n$ and $q \in {}^*X_0 \cup {}^*X_{n-1}$. Consider the internal binary relation $\Phi = \{(x, y) \mid (x \in y \in {}^*F(B)) \wedge (x \in {}^*X_0 \cup {}^*X_{n-1}) \wedge (y \in X_n)\}$. Suppose that $\{(a_1, b_1), \ldots, (a_j, b_j)\} \subset \Phi$. By *-transfer of the standard theorem, it follows that $b = b_1 \cup \cdots \cup b_j \in {}^*F(B)$ and $a_i \in b,\ i = 1, \ldots, j$. Hence, Φ is a concurrent on its domain. But $A \subset D(\Phi)$ and has the appropriate cardinality. Hence, there exists some $\Omega \in {}^*(F(B))$ such that $A \subset \Omega \subset B$. This completes the proof.

Corollary 1.3.1. *Consider standard A. Then there exists a hyperfinite set Ω such that ${}^\sigma A \subset \Omega \subset {}^*A$.*

Proof. Simply note that the cardinality of A is less than κ.

The hyperfinite sets are the basic building blocks of the nonstandard theory of probability spaces.

2. <u>Generalized Functions.</u>

The functions considered are real valued functions. It is not difficult to extend all of the results in this section to complex valued functions. Further all standard functions map $\mathbb{R}$ into $\mathbb{R}$. Let C^∞

be the set of all real valued functions defined on $\mathbb{R}$ which have derivatives of all orders at each $x \in \mathbb{R}$. The set ${}^*C^\infty$ contains some very interesting *-continuous and *-differentiable functions. Throughout this paper, nonempty $\mathcal{D} \subset C^\infty$ is always the notation for what is called the *test space.* Each member of $\mathcal{D}$ must be a function with bounded support. This implies that if $g \in \mathcal{D}$, then there is some $c \in \mathbb{R}$ such that $g(x) = 0$ for all $|x| \geq c$.

Usually one is interested in the generation of linear functionals. The customary generating functions are maps from $\mathbb{R}$ into $\mathbb{R}$. The basic method of generation is by integration. Usually, the customary integration is Lebesgue integration although it appears the generalized Riemann integral can also be used. The reason that Lebesgue is useful is that this integral applies to many highly discontinuous standard functions, has useful convergence properties and, operationally, is sufficient. For our purposes, the Lebesgue integral is considered as an operator in the sense that it is 3-tuple with the first coordinate a function, the second an interval (or for other applications a measurable subset of $\mathbb{R}$), and the third coordinate the value when it exists.

Our customary standard generating functions, CS, have the property that they are Lebesgue measurable on $[a,b]$, for $a \leq b,\ a,b \in \mathbb{R}$ and the integral $\int_a^b (f(x))^2\,dx \in \mathbb{R}$ (i.e. $f \in \mathcal{L}^2([c,d])$ a classical Banach Space). If f is measurable and bounded on $[a,b]$ and $f \in \mathcal{L}([a,b])$ then $f \in \mathcal{L}^2([a,b])$. [7, p. 219] It is known that if $f \in \mathcal{L}^2(E)$, and for the Lebesgue measure, m, $m(E) \in \mathbb{R}$, then $f \in \mathcal{L}(E)$. [7, p. 220] From this, it follows that $\int_{-\infty}^{\infty} f(x)g(x)\,dx \in \mathbb{R}$ for each $g \in \mathcal{D}$. Our functions are restricted to members of internal set $\cap\{{}^*\mathcal{L}^2([c,d]) \mid (c \leq d) \wedge (c,d \in {}^*\mathbb{R})\}$ so that the *-transform of the classical Schwarz inequality applies. For the purposes of this paper, the set of internal generating functions is has a slightly different formation and can only be assumed to be an external subset of $\cap\{{}^*\mathcal{L}^2([c,d]) \mid (c \leq d) \wedge (c,d \in {}^*\mathbb{R})\}$. Recall that $\mathcal{O}$ is the set of all limited numbers in ${}^*\mathbb{R}$. Note that this set is also called the set of finite numbers. It is clear that if f is a customary standard function, then for each $c \leq d,\ c,d \in {}^*\mathbb{R}$, ${}^*\!\int_c^d ({}^*f(x))^2\,dx \in {}^*\mathbb{R}$. Thus ${}^*\!\int_c^d ({}^*f(x))^2\,dx \in {}^*\mathbb{R}$ when $c,d \in \mathcal{O}$. Further, $\int_{-\infty}^{\infty} f(x)g(x)\,dx = r$ implies that ${}^*\!\int_{-\infty}^{\infty} {}^*f(x)\,{}^*g(x)\,dx = {}^*r = r$. First, are there members of CS such that ${}^*\!\int_a^b ({}^*f(x))^2\,dx \in \mathcal{O}$ when $c,d \in \mathcal{O}$?

Theorem 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$. Suppose that f is measurable (in the Lebesgue sense) and bounded on all intervals $[a,b],\ a \leq b,\ a,b \in \mathbb{R}$. Then for each $c \leq d,\ (c,d \in {}^*\mathbb{R})$, ${}^*\!\int_c^d {}^*f(x)\,dx \in {}^*\mathbb{R}$ and ${}^*\!\int_c^d ({}^*f(x))^2\,dx \in {}^*\mathbb{R}$ and if $c,d \in \mathcal{O}$, then ${}^*\!\int_c^d {}^*f(x)\,dx \in \mathcal{O}$ and ${}^*\!\int_c^d ({}^*f(x))^2\,dx \in \mathcal{O}$.*

Proof. It follows from the hypotheses, that $f \in \mathcal{L}([a,b])$ and for each interval $[a,b]$ there exists some $M \in \mathbb{R}$ such that $|f(x)| < M$, for each $x \in [a,b]$. By *-transfer, ${}^*\!\int_c^d {}^*f(x)\,dx \in {}^*\mathbb{R}$ for $c \leq d,\ (c,d \in {}^*\mathbb{R})$. Let $c \leq d$ and $c,d \in \mathcal{O}$. Then there exist $a,b \in \mathbb{R}$ such that $c \approx a,\ d \approx b$. There are four cases to consider but one will suffice as a prototype. Suppose that $c \leq a,\ d \leq b$. Let m denote the Lebesgue measure on the measurable subsets of $\mathbb{R}$. For each $x,y \in {}^*\mathbb{R}$ such that $x \leq y$, ${}^*m([x,y]) = y - x$ by *-transfer. Hence, ${}^*m([c,a]) = c - a \approx 0$, ${}^*m([d,b]) = b - d \approx 0$. Now ${}^*\!\int_c^b {}^*f\,dx = {}^*\!\int_c^a {}^*f\,dx + {}^*\!\int_a^d {}^*f\,dx + {}^*\!\int_d^b {}^*f\,dx$ by *-transfer. Consider ${}^*\!\int_c^a {}^*f\,dx$. There exists some $g \in \mathbb{R}$ such that $g \leq c$. Hence $[c,b] \subset {}^*[g,b]$. There exists some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for each $x \in [c,b]$. Again by *-transfer $|{}^*f(x)| \leq M$ for each $x \in {}^*[c,b]$. By *-transfer,

$$-M({}^*m([c,a])) = -M(a-c) \leq {}^*\!\int_c^a {}^*f\,dx \leq M({}^*m([c,a]) = M(a-c).$$

Consequently, ${}^*\!\int_c^a {}^*f\,dx \approx 0$. In like manner. ${}^*\!\int_d^b {}^*f\,dx \approx 0$. Therefore, ${}^*\!\int_c^d {}^*f\,dx \approx {}^*\!\int_a^b {}^*f\,dx = r \in \mathbb{R}$. Hence, ${}^*\!\int_c^d {}^*f(x)\,dx \in \mathcal{O}$.

For the second part, simply consider the known standard result that if $f \in \mathcal{L}([a,b])$ and g is bounded and measurable on $[a,b]$, then $fg \in \mathcal{L}([a,b])$ and apply the first part. This completes the proof.

Corollary 2.1.1. *Let continuous* $f\colon \mathbb{R} \to \mathbb{R}$. *Then for each* $c \leq d$, $(c, d \in {}^*\mathbb{R})$, ${}^*\!\int_c^d {}^*\!f(x)\,dx \in {}^*\mathbb{R}$ *and* ${}^*\!\int_c^d ({}^*\!f(x))^2\,dx \in {}^*\mathbb{R}$ *and if* $c, d \in \mathcal{O}$, *then* ${}^*\!\int_c^d {}^*\!f(x)\,dx \in \mathcal{O}$ *and* ${}^*\!\int_c^d ({}^*\!f(x))^2\,dx \in \mathcal{O}$.

Proof. Clearly, $f \in \mathcal{L}([a,b])$ and $f \in \mathcal{L}^2([a,b])$. The same proof as theorem 2.1 yields the conclusions.

Definition 2.1 (Generalized Functions) Let T be the set of internal functions such that for each $f \in T$, (i) $f\colon {}^*\mathbb{R} \to {}^*\mathbb{R}$ and (ii) ${}^*\!\int_c^d (f(x))^2\,dx \in {}^*\mathbb{R}$ for each pair $c, d \in \mathcal{O}$, $c \leq d$, and (iii) ${}^*\!\int_{-\infty}^{\infty} f(x)\,{}^*g(x)\,dx \in \mathcal{O}$ for each $g \in \mathcal{D}$.

A function f such that f^2 is a limited integral over limited intervals, will be said to have the *limited* (ii) property.

Theorem 2.2. *Suppose that internal* $f\colon {}^*\mathbb{R} \to {}^*\mathbb{R}$, *has the limited (ii) property, then* $f \in T$.

Proof. From the hypotheses, f satisfies (i) and (ii) of Definition 2.1. We need only show that f satisfies (iii). We know that for $g \in \mathcal{D}$ there is some $c \in \mathbb{R}^+$ such that $g(x) = 0$ for all $x \in \mathbb{R}$ such that $|x| \geq c$. Then by Schwarz's inequality (in concise notation)

$$\left({}^*\!\int_{-\infty}^{\infty} f\,{}^*g\right)^2 = \left({}^*\!\int_{-c}^{c} f\,{}^*g\right)^2 \leq$$

$$\left({}^*\!\int_{-c}^{c} f^2\right)\left({}^*\!\int_{-c}^{c} {}^*g^2\right).$$

But since g is continuous on $[-c, c]$, $\left(\int_{-c}^{c} g^2\right) = r \in \mathbb{R}$ implies that $\left({}^*\!\int_{-c}^{c} {}^*g^2\right) = r \in \mathcal{O}$. Since f has the limited (ii) property, then

$$\left({}^*\!\int_{-c}^{c} f^2\right)\left({}^*\!\int_{-c}^{c} g^2\right) = h \in \mathcal{O}.$$

Consequently, ${}^*\!\int_{-c}^{c} f\,{}^*g \in \mathcal{O}$ and the proof is complete.

Example 2.1. Does T contain nonextended standard functions? Let $0 \neq \epsilon \approx 0$ (i.e. $\epsilon \in \mu(0)$, where $\mu(0)$ is the set of infinitesimals.) Define the function $f = \{(x, y) \mid (x \in {}^*\mathbb{R}) \wedge (y \in {}^*\mathbb{R}) \wedge (y = \epsilon)\}$. From the internal definition principle [3], f is an internal *-constant function. It is not the extension of a standard function since $\epsilon \notin \mathbb{R}$. Moreover, ${}^*\!\int_c^d f^2 = \epsilon^2(d - c) \in \mu(0) \subset \mathcal{O}$ for all c, d, $(c \leq d)$ such that $c, d \in \mathcal{O}$. By Theorem 2.2, $f \in T$. Also consider the internal function $f_1 = \{(x, y) \mid (x \in {}^*\mathbb{R}) \wedge (y \in {}^*\mathbb{R}) \wedge (y = {}^*\!\sin(x + \epsilon))\}$. Now f_1^2 is *-continuous on ${}^*\mathbb{R}$ and, hence, *-integrable on any $[c, d]$, $(c \leq d)$, $c, d \in \mathcal{O}$. Further, $-1 \leq {}^*\!\sin^2(x + \epsilon) \leq 1$ for all $x \in {}^*\mathbb{R}$. Hence, $-1(d - c) \leq {}^*\!\int_c^d {}^*\!\sin^2 \leq 1(d - c)$ for all limited $c \leq d$ implies that ${}^*\!\int_c^d {}^*\!\sin^2 \in \mathcal{O}$ for all limited $c \leq d$. Again by Theorem 2.2, $f_1 \in T$.

Theorem 2.3 *The set* ${}^\sigma(CS) \subset T$.

Proof. From the discussion prior to Theorem 2.1.

From Theorem 2.1, Corollary 2.1.1, and by Theorem 2.2, T contains many significant extended standard functions, among others, that will be useful later in this investigation. The next result also indicates that one of the conclusions is sufficient for an internal function to be a member of T.

The next definition indicates why (iii) in Definition 2.1 is significant.

Definition 2.2 (The Quasi-Inner Product) Let $f \in T$ and $g \in \mathcal{D}$. Define $\langle f, {}^*g\rangle = {}^*\!\int_{-\infty}^{\infty} f(x)\, {}^*g(x)\, dx$. Note that $\langle \cdot, \cdot \rangle$ is an ordered pair notation since the elements come from possibly different sets.

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3. Some Abstract Algebra.

For most applications of the theory of generalized functions, it may be assumed or a function can be appropriately redefined so that the standard function being considered is at the least bounded on the closed intervals. Call $[c, d]$, where $c \leq d$ and $c, d \in {}^*\mathbb{R}$ a *limited *- closed interval.* Recall that if $f \in \mathcal{L}^2(E)$ for measurable E such that $m(E) \in \mathbb{R}$, then $f \in \mathcal{L}(E)$. [7, p. 220] For the internal functions that concern us, *-transfer says that if $f \in {}^*\mathcal{L}^2([c, d]), (c \leq d,\ c, d \in {}^*\mathbb{R})$, then $f \in {}^*\mathcal{L}[c, d]$. This last statement certainly holds for the limited *-closed intervals. Recall the standard result that if $f \in \mathcal{L}([a, b])$, and g is measurable and bounded on $[a, b]$, then $fg \in \mathcal{L}([a, b]$ [7, p. 219]. Thus by *-transfer for internal function f that is *-bounded on a limited *-closed interval $[c, d]$, the function $f \in {}^*\mathcal{L}^2([c, d])$ if and only if $f \in {}^*\mathcal{L}([c, d])$. As shown in the next theorem, in some special cases, the set T is closed under multiplication of functions.

Theorem 3.1 *The following algebraic properties hold.*

(a) The set T is an unitary $\mathcal{O}$- module (left and right) over the set of limited numbers $\mathcal{O}$ and T is a linear space over $\mathbb{R}$.

*(b) If $f, h \in T$ and f, h are *-bounded in limited *-closed intervals and the product fh has the limited (ii) property, then $fh \in T$.*

(c) For the set of all continuous functions defined on $\mathbb{R}$, $C(\mathbb{R})$, ${}^\sigma C(\mathbb{R}) \subset T$

(d) The set ${}^\sigma\mathcal{D}$ is an ideal in ring with unity ${}^\sigma C^\infty$.

(e) If $f \in T$ and ${}^*g \in {}^\sigma C^\infty$, then $f\, {}^*g \in T$.

(f) The the real valued operator $\langle \cdot, \cdot \rangle$ is linear with respect to the field $\mathbb{R}$ in the first and second coordinates. The standard part of $\langle \cdot, \cdot \rangle$ is an inner product on ${}^\sigma\mathcal{D}$.

Proof.

(a) From the *-transfer of the known properties of $\mathcal{L}^2([c, d])$, T is closed under function addition. Since $\mathcal{O}$ is a ring, if $\lambda \in \mathcal{O}$, then $\lambda f \in T$. The functions ${}^*\mathbf{1} \equiv 1$, ${}^*\mathbf{0} \equiv 0 \in T$. Hence, T is a unitary $\mathcal{O}$-module over the ring $\mathcal{O}$.

(b) A standard result says that if $f \in \mathcal{L}^2([a, b])$ and $g \in \mathcal{L}^2([a, b])$, then $fg \in \mathcal{L}([a, b])$. By *-transfer, this statement holds for the limited *-closed intervals. If internal f and internal g are *-bounded on a limited *-closed interval, then internal fg is *-bounded on a limited *-closed interval. From our discussion prior to the statement of Theorem 3.1, in this case, $fg \in {}^*\mathcal{L}([c, d])$. But the *-measurable internal function fg is *-bounded on limited *-closed intervals. Thus $fg \in {}^*\mathcal{L}^2([c, d])$ for the limited *-closed intervals. Further, $(fg)^2$ is *-bounded on $[c, d]$. Obviously, fh satisfies (i) and, from the hypothesis, (ii) of Definition 2.1. Now by Theorem 2.2, it follows that $fh \in T$.

(c) ${}^\sigma C(\mathbb{R}) \subset {}^\sigma(CS) \subset T$.

(d) The sum and product of members of C^∞ with bounded support have bounded support and $\mathbf{0} \in \mathcal{D}$. Hence ${}^\sigma\mathcal{D}$ is a subring of ${}^\sigma C^\infty$. From the bounded support property, if $h \in C^\infty$ and $g \in \mathcal{D}$, then gh has bounded support. Thus ${}^\sigma\mathcal{D}$ is an ideal in ${}^\sigma C^\infty$.

(e) Suppose you have a standard function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \in \mathcal{L}^2([a,b])$ on standard interval $[a,b]$. Since $h^2 \in C^\infty$ is bounded and measurable on $[a,b]$, $(fh)^2 \in \mathcal{L}([a,b])$ by Theorem 22.4s in [6, p. 127.]. Thus, by *-transfer, for $f \in T$ and ${}^*h \in {}^\sigma C^\infty$, $f\,{}^*h$ satisfies (i) and (ii) of Definition 2.1. From (d), if $g \in \mathcal{D}$, then $hg \in \mathcal{D}$. Consequently, ${}^*\!\int_{-\infty}^{\infty} f\,{}^*h\,{}^*g \in \mathcal{O}$ for each $g \in \mathcal{D}$.

(f) Only the basic algebra for members of T, where it is defined, needs to be verified. Let $f \in T$, $\lambda \in \mathbb{R}$. We know that if $f, h \in T$, $f + h \in T$ and λf, $\lambda h \in T$. If ${}^*g_1, {}^*g_2 \in {}^\sigma\mathcal{D}$, ${}^*g_1 + {}^*g_2 \in {}^\sigma\mathcal{D}$ and $\lambda\,{}^*g_1$, $\lambda\,{}^*g_2 \in {}^\sigma\mathcal{D}$. For the second coordinate, $\lambda\langle f, {}^*g_1 + {}^*g_2)\rangle = \lambda\,{}^*\!\int_{-\infty}^{\infty} f({}^*g_1 + {}^*g_2) = {}^*\!\int_{-\infty}^{\infty} \lambda f\,{}^*g_1 + \lambda f\,{}^*g_2 = {}^*\!\int_{-\infty}^{\infty} \lambda f\,{}^*g_1 + {}^*\!\int_{-\infty}^{\infty} \lambda f\,{}^*g_2 = {}^*\!\int_{-\infty}^{\infty} \lambda f\,{}^*g_1 + {}^*\!\int_{-\infty}^{\infty} \lambda f\,{}^*g_2 = \lambda\langle f, {}^*g_1\rangle + \lambda\langle f, {}^*g_2\rangle$. In like manner for the first coordinate and, from the above, $\lambda\langle f, g\rangle = \langle \lambda f, g\langle = \langle f, \lambda g\rangle$. The standard part operator is linear over $\mathbb{R}$. Hence, the composition of $\langle\cdot,\cdot\rangle$ and $\mathtt{st}(\cdot)$ is linear over $\mathbb{R}$. Moreover, this composition yields a member of $\mathbb{R}$. Now $\langle\cdot,\cdot\rangle$ is defined on all members of ${}^\sigma\mathcal{D}$ independent of order. Further, if $g, h \in {}^\sigma\mathcal{D}$, then $\langle g, h\rangle = \langle h, g\rangle \in \mathbb{R}$ implies that $\mathtt{st}(\langle g, h\rangle) = \mathtt{st}(\langle h, g\rangle)$, and $\langle g, g\rangle \geq 0$ implies that $\mathtt{st}(\langle g, g\rangle \geq 0)$. Since $\mathcal{D}$ contains only continuous functions (with bounded support), $\langle g, g\rangle = 0$ if and only if $g = \mathbf{0}$. Thus on ${}^\sigma\mathcal{D}$ the operator $\mathtt{st}(\langle\cdot,\cdot\rangle)$ is an inner product.

4. Functionals on T.

Definition 4.1 (The Functional) Let fixed $f \in T$. Then, for each ${}^*g \in {}^\sigma\mathcal{D}$, define $f[g] = \mathtt{st}(\langle f, {}^*g\rangle)$. Let $T_0 = \{f \mid (f \in T) \land \forall x((x = {}^*g) \land ({}^*g \in {}^\sigma\mathcal{D}) \to (\mathtt{st}(\langle f, x\rangle) = 0))\}$.

From the fact that the standard part operator is linear over $\mathbb{R}$, it follows from Theorem 3.1 (f), that $f[\cdot]$ is a linear functional on${}^\sigma\mathcal{D}$. Let $\mathcal{F} = \{f[\cdot] \mid f \in T\}$. Obviously there is a surjection $\Phi\colon T \to \mathcal{F}$. Since T is a linear space over $\mathbb{R}$ and $\Phi(\mathbf{0}) = \mathbf{0}[g] = 0$ for each $g \in {}^\sigma\mathcal{D}$ and preserves scalar products and "sums," Φ is a vector space homomorphism.

Theorem 4.1. *The function $f \in T_0$ if and only if $\langle f, {}^*g\rangle \approx 0$ for all $g \in \mathcal{D}$.*

Proof. This comes from the fact that $\mathtt{st}(a) = 0$ if and only if $a \approx 0$.

Theorem 4.2 *If $f \in T$, and ${}^*\!\int_c^d (f(x))^2\,dx \approx 0$ for all limited*-closed intervals, then $f \in T_0$.*

Proof. This is the same as the proof of Theorem 2.2.

Obviously, more than one $f \in T$ can yield the zero functional and $T_0 \neq \emptyset$ since $\mathbf{0} \in T_0$. Now T_0 is the kernel for this homomorphism, and, as is-known, the quotient linear space T/T_0 is isomorphic to $\mathcal{F}$. Each element in T/T_0 is an equivalence class of members of T. Then for $h, g \in T$, it follows that $f, h \in \alpha \in T/T_0$ if and only if $h - f \in T_0$ if and only if $h[g] = f[g] = 0$ for all ${}^*g \in {}^\sigma\mathcal{D}$. It is this isomorphism that allows us to correspond a subset of T/T_0 to all of the Schwarz generalized functions.

Example 4.1. Does T contain functions that yield the Dirac property? Let

$$b(t) = \begin{cases} \exp(-1/(1-t^2)), & |t| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

This is a version of Cauchy's flat function and it is known that $b \in \mathcal{D}$. We now compress this function. Let $0 < \epsilon \in \mu(0)$ (a positive infinitesimal). Let $c(t) = b(t/\epsilon)$. By *-transfer, $c \in {}^*C^\infty$ with support $[-\epsilon, \epsilon]$. We can normalize c by letting $k = {}^*\!\int_{-\infty}^{\infty} c(t)\, dt \neq 0$, and writing $d(t) = (1/k)c(t)$. Obviously, $d \in {}^*C^\infty$, is nonnegative, and ${}^*\!\int_{-\infty}^{\infty} d(t)\, dt = 1$. First, we show that $d \in T$. By *-transfer, d is *-bounded *-measurable on each $[c, d],\ c, d \in {}^*\mathbb{R}$. Hence $d \in {}^*\mathcal{L}^2([c, d]),\ c, d \in \mathcal{O}$ (i.e. $d \in {}^*(CS)$) and satisfies (i) and (ii) of Definition 2.1. We need only show that for all ${}^*g \in {}^\sigma\mathcal{D}$, $\langle d, {}^*g\rangle \approx {}^*g(0) = g(0)$. Recall that the operators ${}^*\max$ and the max on ${}^*\mathbb{R}$ are the same operator. By *-transfer of the standard theorem,

$$|{}^*\!\int_{-\infty}^{\infty} d\, {}^*g| \leq \sup\{|{}^*g(t)| \mid t \in [-\epsilon, \epsilon]\} \cdot {}^*\!\int_{-\epsilon}^{\epsilon} d.$$

Note that ${}^*\!\int_{-\epsilon}^{\epsilon} d = 1$. The function $|g|$ is continuous at 0. Hence, for each $t \in \mu(0)$, $|{}^*g(t)| \approx g(0)$. From the *-transfer of the extreme value theorem, there exists some $t_1 \in [-\epsilon, \epsilon] \subset \mu(0)$ such that $\sup\{|{}^*g(t)| \mid t \in [-\epsilon, \epsilon]\} = |{}^*g(t_1)|$ (i.e. sup = max). Hence, $\sup\{|{}^*g(t)| \mid t \in [-\epsilon, \epsilon]\} \in \mu(g(0)) \subset \mathcal{O}$. Thus $d \in T \cap {}^*(CS)$. In a similar manner, noting that d is nonnegative, we have we have that $g(0) \approx \min\{{}^*g(t) \mid t \in [-\epsilon, \epsilon]\} \leq \langle d, {}^*g\rangle \leq \max\{{}^*g(t) \mid t \in [-\epsilon, \epsilon]\} \approx g(0)$. Consequently, the functional $d[g] = g(0)$ for all ${}^*g \in {}^\sigma\mathcal{D}$. But this last statement is the "shifting" property of Dirac when viewed as a *-Lebesgue integration over ${}^*\mathbb{R}$. The same method shows that for any positive $n \in \mathbb{N}$, $d^n[g] = g(0)$.

There are infinitely many internal functions in ${}^*C^\infty$ that are in T and that determine the Dirac functional. In the standard theory, no such standard function exists and "something" is only symbolically introduced relative to the required shifting property. This yields what are called "singular" generalized functions. From the nonstandard viewpoint, at least for the $d[\cdot]$, such a concept of "singular" is no longer meaningful.

Since T/T_0 is isomorphic to $\mathcal{F}$, then each $f \in T$ such that $f[g] = g(0)$ for all $g \in \mathcal{D}$ are in the same member of T/T_0. We call this the *Dirac delta* equivalence class and denote it by δ. Note that $d^n \in \delta$.

Example 4.2. The set ${}^*\mathcal{D} \not\subset T$. Consider the function b of example 4.1. Then $0 < \inf\{d(t) \mid -1/2 \leq t \leq 1/2\} = d(1/2) \in \mathbb{R}$. Further, $0 < \int_{-\infty}^{\infty} d = r \in \mathbb{R}$. As pointed out, $b \in \mathcal{D}$. Let $\Lambda \in \mathbb{N}_\infty$ (the infinite natural numbers). Then by *-transfer, $\Lambda b \in {}^*\mathcal{D}$. Now $\inf\{\lambda\, {}^*d(t) \mid -1/2 \leq t \leq 1/2\} = \Lambda d(1/2) \in {}^*\mathbb{R} - \mathbb{R}$ and

$$\Lambda d(1/2) \left({}^*\!\int_{-\infty}^{\infty} {}^*d\right) \leq {}^*\langle \lambda\, {}^*d, {}^*d\rangle.$$

Thus ${}^*\langle \lambda\, {}^*d, {}^*d\rangle \notin \mathcal{O}$.

Definition 4.2 (Pre-generalized Functions) Each member α in the quotient linear space T/T_0 is called a pre-generalized function and each member of T is a generalized function. From this point on, lower case Greek letters will always denote pre-generalized functions.

One of the reasons, the set T/T_0 is called the set of pre-generalized functions is that a member of T/T_0 need not correspond to a Schwarz generalized function. But before corresponding pre-generalized functions to Schwarz generalized functions, we have the following remarkable result first proved by Robinson. The functionals in $\mathcal{F}$ are specifically generated by the ${}^*\!\int_{-\infty}^{\infty}$. Does this exhaust the entire collection of all linear functionals defined on ${}^\sigma\mathcal{D}$? The following result shows the power of the enlargement concept.

Theorem 4.3. *Let Δ represent any linear functional defined on ${}^\sigma\mathcal{D}$. Then there exists a *-polynomial $p_\Delta \in {}^*C^\infty \cap T$ such that $\Delta = p_\Delta[\cdot] \in \mathcal{F}$.*

Proof. First, let Π be the set of all polynomials defined on $\mathbb{R}$ and $\Delta\colon \mathcal{D} \to \mathbb{R}$. Note that $\Pi \subset C^\infty$. Consider the binary relation $R = \{((g, \Delta(g)), p) \mid (g \in \mathcal{D}) \wedge (\Delta(g) \in \mathbb{R}) \wedge (p \in \Pi) \wedge (\int_{-\infty}^{\infty} pg = \Delta(g))\}$. What is needed is to show that R is concurrent on the domain $\mathbb{R} \times \mathcal{D}$.

Consider nonempty $\mathcal{D}$ and a nonempty finite linear independent $L = \{g_j \mid (j = 1, \ldots, m) \wedge (1 \leq m)\} \subset \mathcal{D}$. Then there exists some $c > 0$ such that $g_j(x) = 0,\ i \leq j \leq m$, and c can be selected so that each g_j is zero in a neighborhood of $-c$ and c. From this we also have that for any $f \in T$,

$$f[g_j] = \mathtt{st}({}^*\!\int_{-c}^{c} f\, {}^*g_j) = a_j,\ 1 \leq j \leq m. \tag{4.2.1}$$

Since each $g_j \in \mathcal{D} \subset C^\infty$, let each g_j be represented in terms of a series expansion of Legendre polynomials P_i where, using a simple transformation of the independent variable, the P_i have been extended to converge on $[-c, c]$ rather than $[-1, 1]$. Hence, $g_j(x) = \sum_{n=0}^{\infty} a_n^j P_n(x)$, for all $x \in [-c, c]$ and the convergence being uniform on any $[-d, d]$, $0 < d < c$. We now use the method of the infinite matrix, a method used by Robinson and Bernstein to solve a specific case of the invariant subspace problem. From the linear independent assumption, the matrix

$$B = \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ a_0^m & a_1^m & a_2^m & \cdots \end{pmatrix}$$

must be of rank m. Thus there is a finite collect of members B and, hence, of subscripts $0 \leq j_i \leq \cdots \leq j_m$ such that

$$A = \begin{pmatrix} a_{j_1}^1 & a_{j_2}^1 & a_{j_3}^1 & a_{j_m}^1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{j_1}^m & a_{j_2}^m & a_{j_3}^m & a_{j_m}^m \end{pmatrix}$$

and A is nonsingular. Now write

$$A^{-1} \begin{pmatrix} g_1(x) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x) \end{pmatrix} = \begin{pmatrix} h_1(x) \\ \cdot \\ \cdot \\ \cdot \\ h_m(x) \end{pmatrix}.$$

Thus, we can write

$$h_l(x) = P_{j_l}(x) + k_l(x),\ 1 \leq l\ \leq m,$$

where the Legendre polynomials in each $k_l(x)$ do not contain any of the $P_{j_l}(x),\ 1 \leq l \leq m$.

Now let

$$A^{-1}\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}.$$

Consider the polynomial $p(x) = c_1P_{j_1} + \cdots c_mP_{j_m}(x)$. We want to obtain the proper $c_1, \ldots, c_m$ such that $p[g_j] = a_j,\ 1 \leq j \leq m$. First, note that from the orthogonality of the Legendre polynomials $\int_{-c}^{c} pk_l = 0,\ 1 \leq l \leq m$ and we know that $\int_{-c}^{c} P_n^2 = r_n \neq 0$. Select $c_l = (1/r_l)b_l,\ 1 \leq l \leq m$. Substituting this p with these coefficients into (4.2.1) yields

$$p[g_j] = \mathtt{st}({}^*\!\!\int_{-c}^{c} {}^*\!p\, {}^*\!g_j) = \int_{-c}^{c} pg_j = a_j,\ 1 \leq j \leq m. \tag{4.2.2}$$

Since every member of $\mathcal{D} - L$ (if any) is a linear combination of the members of L, then it follows from the linearity of Δ that for each $g \in \mathcal{D},\ p[g] = \Delta(g)$. Hence, the relation R is concurrent on $\mathbb{R} \times \mathcal{D}$ and, hence on ${}^\sigma\mathbb{R} \times {}^\sigma\mathcal{D}$. Thus, there exists some $p_\Delta \in {}^*\Pi \subset {}^*C^\infty$ such that for all $(g, \Delta(g)) \in \mathbb{R} \times {}^\sigma\mathcal{D},\ ((g, \Delta(g)), p_\Delta) \in {}^*\!R$. Since $\mathbb{R} \subset \mathcal{O},\ p_\Delta \in T$ and ${}^*\!\!\int_{-\infty}^{\infty} p_\Delta\, {}^*\!g = \Delta(g)$. Consequently, $p_\Delta[g] = \Delta(g)$ and the proof is complete.

Theorem 4.3 is remarkable, since if $\alpha \in T/T_0$, there is $p_\Delta \in {}^*\Pi$ such that $p_\Delta \in \alpha$. Furthermore, p_Δ is a *-finite sum of *-Legendre polynomials. This means, as difficult as it might be to imagine, there is a $p_\Delta \in \delta$ such that $p_\Delta[g] = g(0)$ for each ${}^*\!g \in {}^\sigma\mathcal{D}$. The internal function d used in Example 4.1 is NOT a member of ${}^*\Pi$ by *-transfer of the standard properties of the standard function $c(t/a),\ 0 < a$. The function p_Δ that generates every linear functional is also a member of T_R. Of course, Theorem 4.3 holds for other functions as well that are either simple modifications of the P_i or are such things as finite sums of trigonometric functions. **Note that every linear functional on ${}^\sigma\mathcal{D}$ (i.e. $\mathcal{D}$) can be generated by the Definition 4.1 process by means of *-Riemann integration.** Although it is possible to remove many functions from each pre-generalized function α by considering the quotient group formed from the sets $P_0 = T \cap {}^*C^\infty$, (or $T \cap {}^*\Pi$) and $p_1 = T_0 \cap {}^*C^\infty$, (or $T_0 \cap {}^*\Pi$), it is more significant to have such functions as $d \in \delta$. Later for a necessary simplification process, we will call a generalized function the function in ${}^*C^\infty$ that exists by Theorem 4.3 and not consider the equivalence class at all.

5. Schwarz Generalized Functions.

Definition 5.1 (Schwarz Generalized Functions) A $f[\cdot] \in \mathcal{F}$ is a *Schwarz generalized function* if given any sequence $\{g_n\} \subset \mathcal{D}$ such that

(i) there exists some $a \in \mathbb{R}$ such that for each $n \in \mathbb{N}$, $g_n(x) = 0$ for all $|x| > a$,

(ii) for each natural number $k \geq 0$, $g_n^{(k)}(x) \to \mathbf{0}$ uniformly on $[-a, a]$,

(iii) then $f[g_n] \to 0$.

Let $\mathcal{D}'$ denote the set of Schwarz generalized functions.

For every $f[\cdot] \in \mathcal{D}'$, there exists unique $\alpha \in T/T_0$ under our isomorphism. Such an α is called $\mathcal{D}'$-pre-generalized function and an $f \in \alpha$ an $\mathcal{D}'$-generalized function. [The term *Schwarz generalized function* means the linear functions.] As previously pointed out, ${}^{\sigma}(CS) \subset T$. Using only $k = 0$, properties of Lebesgue integration, the Schwarz inequality, if $f \in CS$, then it follows that ${}^*f[\cdot] \in \mathcal{D}'$. Using the *-transform of these previous properties, it follows, using the internal function d defined in Example 4.1, that $d[\cdot] \in \mathcal{D}'$ although d is not an extended standard function. This shows the advantage of selecting specific members of a pre-generalized function.

Theorem 5.1. *Let $f,\ h \in \alpha$. Suppose that using the *-transform of the definition of the derivative that f and h possess a *-derivative f' and h' for each $x \in {}^*\mathbb{R}$ and that $f',\ h'$ are *-continuous for each $x \in {}^*\mathbb{R}$. Then $f',\ h' \in T$ and there exists a β such that $f', h' \in \beta$.*

Proof. By *-transfer of the continuous case, it follows that f' satisfies (i), (ii) of Definition 4.1. By *-integration by parts, $\mathtt{st}(\langle f',\ {}^*g\rangle) = -\mathtt{st}(\langle f,\ {}^*g'\rangle) \in \mathbb{R}$ for each $g \in \mathcal{D}$. Since $g' \in \mathcal{D}$, then $f' \in T$. In like manner, $h' \in T$. Consider $k = f - h$. We know that $\langle k,\ {}^*g\rangle \approx 0$ for each $g \in \mathcal{D}$. Since k satisfies all the *-transformed derivative rules, $\mathtt{st}(\langle k',\ {}^*g\rangle) = -\mathtt{st}(\langle k,\ {}^*g'\rangle) = 0$ for each $g \in \mathcal{D}$. Thus $k' \in T_0$ by our remark after Theorem 4.2. Hence, there exists a unique $\beta \in T/T_0$ such that $f',\ h' \in \beta$. This completes the proof.

Corollary 5.1.1 *For every $f[\cdot] \in \mathcal{F}$, there exists a $f'[\cdot] \in \mathcal{F}$ such that $f'[g] = -f[g']$.*

Since every pre-generalized function α contains an internal f such that there is an internal function f' which is the *-continuous *-derivative of f on ${}^*\mathbb{R}$, then to every pre-generalized function there corresponds a unique β such that $f' \in \beta$. We denote such a pre-generalized function by the notation α'.

Theorem 5.2. *For every $k \in \mathbb{N}$, and every α, there exists a unique $\alpha^{(k)} \in T/T_0$.*

Proof. Theorem 4.3 shows that for every $\alpha \in T/T_0$ there exists an $p \in {}^*C^{\infty}$ such that $p \in \alpha$. The result follows by induction and Theorem 5.1.

Corollary 5.2.1 *For every $0 \leq k \in \mathbb{N}$, and for every $f[\cdot] \in \mathcal{F}$, there exists a $f^{(k)}[\cdot] \in \mathcal{F}$ such that $f^{(k)}[g] = (-1)^k f[g^{(k)}]$.*

One can now see why included in the definition of the Schwarz generalized function is the additional part of (ii) for each $k > 0$. For, from above, we have the following important result.

Theorem 5.3. *If α is a $\mathcal{D}'$-pre-generalized function, then $\alpha^{(k)}$ is a $\mathcal{D}'$-pre-generalized function for each $k \in \mathbb{N}$.*

Corollary 5.3.1. *If $f[\cdot] \in \mathcal{D}'$, then, for each $k \in \mathbb{N}$, there exists a unique $f^{(k)}[\cdot] \in \mathcal{D}'$ such that $f^{(k)}[g] = (-1)^k f[g^{(k)}]$.*

Although Corollary 5.3.1 is an important Schwarz generalized function result, the nonstandard theory is more general in that Corollary 5.2.1 holds.

6. "Continuity."

Obviously, the definition of a Schwarz generalized function is designed to give the linear functional a type of continuity. In nonstandard analysis, there are various types of continuity.

For two topological spaces (X, τ) and $(Y, \mathcal{T})$, you always have the concept of a function $f\colon X \to Y$ as being continuous at $p \in X$ if for each $G_1 \in \tau_1$, such that $f(p) \in G_1$, there exists a $G \in \tau$ such that $p \in G$ and $f(G) \subset G_1$. In general, for standard ${}^*\!p \in {}^\sigma X$, the *topological monad* of ${}^*\!p$ for a given topology $\mathcal{T}$ is $\mu_{\mathcal{T}}({}^*\!p) = \cap\{{}^*G \mid p \in G \in \mathcal{T}\}$. Then it can be shown that f is continuous at $p \in X$ if and only if ${}^*f(\mu_\tau({}^*\!p)) \subset \mu_{\mathcal{T}}({}^*f({}^*\!p))$. Note that the reason we need to use the standard elements in the form ${}^*\!p$ is that it is not assumed that $X \cup Y$ are atoms within our set-theory. Let ${}^*\mathbb{R}^+$ denote the set all positive hyperreal numbers.

Definition 6.1 (Pseudo-metric Generated Space) Given an internal set X and PM_X the internal set of all pseudo-metrics defined on X. If internal map $\lambda \in PM_X$, then for each $x,\ y,\ z \in X$, (i) $\lambda(x,y) \in {}^*\mathbb{R}^+$, (ii) $\lambda(x,y) = \lambda(y,x)$, (iii) if $x = y$, then $\lambda(x,y) = 0$, and (iv) $\lambda(x,z) \leq \lambda(x,y) + \lambda(y,z)$. Let nonempty $\Lambda \subset PM_X$. The entity (X, Λ) is an *pseudo-metric generated space.*

Each space (X, Λ) satisfies the *-transform of any general property for a pseudo-metric. To see this, note that there is some standard X_n such that $X \in {}^*X_n$ and some standard set X_p such that internal $PM_X \in {}^*X_p$. Now, in general, for each $X \in X_n$, there exists a standard set PM_X. Thus there exists a standard set $\mathcal{P} = \{PM_X \mid (PM_X \in X_p) \wedge (X \in X_n)\}$. The internal sets of definition 6.1 are members of ${}^*X_n$ and the internal $PM_X \in {}^*\mathcal{P}$. The defining property for members of internal PM_X is but the *-transform of the standard definition. Hence, using these sets, any general bounded first-order property about standard pseudo-metrics holds, by *-transfer, for members of an internal PM_X. For example, suppose that internal $\lambda \in \Lambda$ is determined by an internal semi-norm $\|\cdot\|_\lambda$ defined on an internal space X linear over ${}^*\mathbb{R}$. Then, for $x, y \in X$, we have that $|\ \|x\|_\lambda - \|y\|_\lambda\ | \leq \|x - y\|_\lambda \leq \|x\|_\lambda + \|y\|_\lambda$. Of course, our basic examples are the standard pseudo-metrics on a standard X.

Example 6.1 Let SM be the set of all internal semi-norms defined on an internal linear space X. Thus $\|\cdot\| \in SM$ if and only if for each $x,\ q \in X$, (i) $\|q\| \in {}^*\mathbb{R}^+$, (ii) for each $\lambda \in {}^*\mathbb{R}$, $\|\lambda q\| = |\lambda|\,\|q\|$, (iii) $\|x+q\| \leq \|x\|+\|q\|$. Then defining internal $\lambda\colon X \times X \to {}^*\mathbb{R}$ by $\lambda(x,q) = \|x-q\|$ gives $\lambda \in PM_X$.

Definition 6.2. (Monads about $q \in X$) For (X,Λ) and $q \in X$ the *monad about* q is $\mu_\Lambda(q) = \{x \mid (x \in X) \wedge \forall\lambda((\lambda \in \Lambda) \to (\lambda(x,q) \in \mu(0)))\}$, where $\mu(0)$ is the set of infinitesimals in ${}^*\mathbb{R}$.

Let internal $\lambda \in \Lambda$. Then the λ-monad about q is $\mu_\lambda(q) = \{x \mid (x \in X) \wedge (\lambda(x,q) \in \mu(0))\}$. It is an important fact that $\mu_\Lambda(q) = \cap\{\mu_\lambda(q) \mid \lambda \in \Lambda\}$.

Example 6.2 Let $\mathcal{S}$ be a standard collection of pseudo-metrics defined on standard X. Consider the usual topology $\mathcal{T}$, generated by the subbase $\mathcal{B}$ of all open balls determined by all the members of $\mathcal{S}$ (i.e. $B(p,\lambda,\epsilon),\ p \in X,\ \lambda \in \mathcal{S},\ \epsilon \in \mathbb{R}^+$). For a topological space, a topological monad, $\mu_{\mathcal{T}}({}^*p)$, about standard $p \in X$ is the set $\cap\{{}^*G \mid p \in G \in \mathcal{T}\} = \cap\{{}^*G \mid {}^*p \in {}^*G \in {}^\sigma\mathcal{T}\}$ and, in general, is equal to $\cap\{{}^*G \mid p \in G \in \mathcal{B}\}$ for any subbase $\mathcal{B}$ for the topology. Thus under Definition 6.2, for $p \in X$, where $\Lambda = {}^\sigma\mathcal{S}$, the monad about p, $\mu_\Lambda({}^*p)$ is topological.

Definition 6.3. ($\approx$ and Monads) When an infinitesimal relation $\approx$ is defined on an internal X, this relation is, usually, an equivalence relation and is used to define a monad about each $q \in X$. The monad about $q \in X$ is the equivalence class $m_\approx(q) = \{x \mid x \approx q\}$.

In general, the monad defined by 6.3 need not be the same as a monad as defined by a topology. But for Definition 6.2, an obvious equivalence relation does exist that correspond these monad concepts.

Definition 6.4 For the space (X,Λ) and any $x,\ y \in X$, let $x \approx y$ if and only if $\forall\lambda((\lambda \in \Lambda) \to (\lambda(x,q) \in \mu(0)))$. Also, for each $\lambda \in \Lambda$, $x \overset{\lambda}{\approx} y$ if and only if $\lambda(x,q) \in \mu(0)$.

It is immediate that the $\approx$ [resp. $\overset{\lambda}{\approx}$] of Definition 6.4 is an equivalence relation on $X \times X$, and that $x \approx y$ [resp. $x \overset{\lambda}{\approx} y$] if and only if $x \in \mu_\Lambda(y)$ [resp. $x \in \mu_\lambda(q)$] if and only if $x \in m_\approx(y)$ [resp. $x \in m_{\overset{\lambda}{\approx}}(y)$.]

Definition 6.5. (S-continuous) For spaces (X,Λ) and (Y,Π) an internal $f\colon X \to Y$ is *S-continuous* at $q \in X$ if $f(\mu_\Lambda(q)) \subset \mu_\Pi(f(q))$. For pseudo-metric space (X,λ) and (Y,π), S-continuity is defined for the spaces $(X,\{\lambda\})$ and $(X,\{\pi\})$.

For an pseudo-metric, λ defined on internal X, you define, for each $\epsilon \in {}^*\mathbb{R}^+$, and for each $q \in X$, (in the usual way) the ball about q as $B(q,\lambda,\epsilon) = \{x \mid (\lambda(x,q) < \epsilon\}$. Another type of continuity, that is usually restricted to standard spaces, is *-continuity. For a standard gauge space (i.e. the topological space generated by the set of pseudo-metrics Λ), then the topology is generated by taking as a bases the finite intersection of standard balls. Since any finite set of real numbers has a minimum, using the neighborhood bases about a standard point, we have that for standard $(X, \Lambda),\ (Y, \Pi)$ a standard functions $f\colon X \to Y$ is continuous at $p \in X$ if and only if for each $\epsilon \in \mathbb{R}^+$ and each $\pi \in \Pi$ there exists a $\delta \in \mathbb{R}^+$ and a finite set of pseudo-metrics $\lambda_i \in \Lambda,\ 1 \leq i \leq n$ such that whenever $x \in X$ and $\lambda_i(x, p) < \delta$ for each $i,\ 1 \leq i \leq n$ then $\pi(f(x), f(p)) < \epsilon$. The *-transfer of this statement is used to define another type of continuity.

Definition 6.6 (*-continuity) Consider pseudo-metric generated spaces $(X, \Lambda),\ (Y, \Pi)$. An internal map $f\colon (X, \Lambda) \to (Y, \Pi)$ is **-continuous* at $q \in X$ if for each $\epsilon \in {}^*\mathbb{R}^+$ and each $\pi \in \Pi$ there exists a $\delta \in {}^*\mathbb{R}^+$ and a *-finite set $\{\lambda_i \mid 1 \leq i \leq \omega\} \subset \Pi$ such that whenever $x \in X$ and $\lambda_i(x, q) < \delta$ for each i such that $1 \leq i \leq \omega$ then $\pi(f(x), f(q)) < \epsilon$. For internal metric spaces, (X, λ) and (Y, π), *-continuity is defined for $(X, \{\lambda\})$ and $(Y, \{\pi\})$.

Notice that *-continuity is not defined solely in terms of the standard points in X. Also the specific ϵ and δ required for *-continuity are members of ${}^*\mathbb{R}^+$, not just the standard positive reals. For standard (X, Λ), (Y, Π), $f\colon (X, \Lambda) \to (Y, \Pi)$ is continuous at $p \in X$ if and only if ${}^*f\colon ({}^*X, {}^*\Lambda) \to ({}^*Y, {}^*\Pi)$ is *-continuous at ${}^*p$.

Example 6.4. By *-transfer, for any infinite $\Lambda \in \mathbb{N}_\infty$, the function $f(x) = {}^*\sin(\Lambda x)$ is *-continuous on ${}^*\mathbb{R}$ with respect to the standard norm $|\cdot|$.

The *-continuous functions have an important place in theoretical physics [4] [See example 6.6 below]. But since they have all of the properties of the continuous functions, they are not considered "interesting" to some members of the mathematics community.

Example 6.5. Although the *-continuous function of example 6.4 is *-continuous at $x = 0$, it is S-discontinuous at $x = 0$. Let infinitesimal $\epsilon = (1/\Lambda)(\pi/2)$. Then for S-continuity we must have that ${}^*\sin(\Lambda \cdot 0) = {}^*\sin(0) = 0 \approx {}^*\sin(\Lambda((1/\Lambda)(\pi/2)) = 1$; a contradiction.

Example 6.6. The function d of example 4.1 that generates the Dirac functional is *-continuous at $x = 0$ but is S-discontinuous. Since the defining statement for d is the *-transform of the collection of all functions c formed by letting $t \to t/a$, where a is any nonzero real number, it follows that d is *-continuous at all $x \in {}^*\mathbb{R}$ (i.e. d is a member of a set of *-continuous functions with this property.) However, $[-\epsilon, \epsilon] \subset d(\mu(0))$ and $d[-\epsilon, \epsilon] = {}^*[0, e^{-1}]$ imply that d is not S-continuous at $x = 0$.

The "S" in S-continuous means "standardly" in the sense that the approximating numbers $\epsilon,\ \delta$ are standard numbers. One example shows what S-continuity is trying to accomplish.

Example 6.7. Let $0 < \epsilon \in \mu(0)$ and $a \in \mathbb{R}$. Define on ${}^*\mathbb{R}$

$$f(x) = \begin{cases} \epsilon + a, & x < 0 \\ a, & x \geq b \end{cases}$$

This is, by *-transfer, an internal function that is *-continuous for all nonzero $x \in {}^*\mathbb{R}$ and is *-discontinuous at $x = b$. But, f is S-continuous at $x = b$. For take any positive $c \in \mathbb{R}$, then no matter what $x \in {}^*\mathbb{R}$ you select, $|f(x) - a| < c$. **That is the *-discontinuity is so "small" that it is not "visible" in the standard world.**

For the topological spaces used in the theory of generalized functions, does the concept of S-continuous correspond to the concept described in Example 6.7? In order to examine the relation between S-continuity and *-continuity for generalized functions, a slight diversion is necessary

For a given standard X, suppose $\mathcal{V}$ is a collection of subsets of X such that $\emptyset \notin \mathcal{V}$ and $\mathcal{V}$ has the finite intersection property (i.e. the intersection of finitely many members is not the empty set). Then the collection $\mathcal{V}$ is a *filter subbase* on X. Further, if there exists some $q \in {}^*X$ such that $q \in {}^*G$ for each $G \in \mathcal{V}$, then $\mathcal{V}$ is, obviously, a filter subbase which is termed the *local filter subbase* at q and denoted by $\mathcal{V}_q$.

Definition 6.7 (Monads of a Filter Subbase) For standard X, let $\mathcal{V}$ be a filter subbase (either local or otherwise) on X. Then let $\mu_{\mathcal{V}} = \cap\{{}^*V \mid V \in \mathcal{V}\}$. If $\mathcal{V}$ is local at $q \in {}^*X$, then since $q \in \mu_{\mathcal{V}}$ the monad is written as $\mu_{\mathcal{V}}(q)$.

Example 6.8 For any standard pseudo-metric space (X, Λ), and $p \in X$, consider the set, $\mathcal{B} = \{B(p, \lambda, \epsilon) \mid (\lambda \in \Lambda) \wedge (\epsilon \in \mathbb{R}^+)\}$, of all balls about p. Then $\mathcal{B}$ is a local filter subbase at p. For any filter subbase $\mathcal{B}$, let $\langle\mathcal{B}\rangle$ be the set obtained by taking finite intersections of members of $\mathcal{B}$. Obviously, $\mathcal{B} \subset \langle\mathcal{B}\rangle$ and $\mu_B = \mu_{\langle\mathcal{B}\rangle}$.

Theorem 6.2. *Let X be standard set and $\mathcal{V}$ any standard filter subbase on X. Then $\mu_{\mathcal{V}} \neq \emptyset$.*

Proof. We know that there is some $X_n,\ n \leq 1$, such that $\mathcal{V} \in X_n$, and if $x \in \mathcal{V}$, then $x \in X_n$. Further, if $y \in x$, then $y \in X_0 \cup X_{n-1}$. Consider the bounded binary relation

$$\Phi(x, y) = \{(x, y) \mid (y \in X_0 \cup X_{n-1}) \wedge (x \in X_n) \wedge (y \in x) \wedge (x \in \mathcal{V})\}.$$

The domain of Φ is $\mathcal{V}$. Let $\{(a_1, b_1), \ldots (a_n, b_n)\} \subset \Phi$. Since $\mathcal{V}$ has the finite intersection property, there exists some $b \in X_0 \cup X_{n-1}$ such that $b \in a_1 \cap \cdots \cap a_n$. Thus $\{(a_1, b), \ldots (a_n, b)\} \subset \Phi$. Hence since we are in an enlargement, there exists $a \in {}^*X_0 \cup {}^*X_{n-1}$ such that $a \in {}^*V$ for each $V \in \mathcal{V}$. Consequently, $\mu_{\mathcal{V}} \neq \emptyset$ and the proof is complete.

Note that if $\mathcal{V}$ is a filter subbase, that the set obtained by taking finite intersections of members of $\mathcal{V}$ does not contain the empty set and is closed under finite intersection.

Theorem 6.3. *Let X be standard set and $\mathcal{V}$ any standard collection of subsets of X, which does not contain the empty set and which is closed under finite intersection. If internal $A \subset \mu_{\mathcal{V}}$, then there exists some *-finite (and, hence, internal) $\Omega \subset {}^*\mathcal{V}$ such that ${}^\sigma\mathcal{V} \subset \Omega$ and $A \subset A_0 = \cap\{B \mid B \in \Omega\} \subset \mu_{\mathcal{V}}$, and $A_0 \in {}^*\mathcal{V}$.*

Proof. First, from Theorem 6.2 $\mu_{\mathcal{V}} \neq \emptyset$. Let $B = \{V \mid (V \in {}^*\mathcal{V}) \wedge (A \subset V)\}$. Then B is an internal subset of ${}^*\mathcal{V}$. Extending Theorem 4.3.4 [3], we know that there exists a *-finite set B_0 such that ${}^\sigma\mathcal{V} \subset B_0 \subset {}^*\mathcal{V}$. Let $\Omega = B \cap B_0$. Since every internal subset of a *-finite set is *-finite, Ω is *-finite. Further, ${}^\sigma\mathcal{V} \subset \Omega$ and $A \subset E$ for each $E \in \Omega$. By *-transfer, ${}^*\mathcal{V}$ is closed under *-finite intersection. Hence $A_0 = \cap\{B \mid B \in \Omega\} \in {}^*\mathcal{V}$, $A \subset A_0$ and $A_0 \subset \mu_{\mathcal{V}}$ and the proof is complete.

Corollary 6.3.1. *Let X be standard set and $\mathcal{V}$ any standard collection of subsets of X, which does not contain the empty set and which is closed under finite intersection. Then $\mu_{\mathcal{V}} = \cup\{G \mid (G \in {}^*\mathcal{V}) \wedge (G \subset \mu_{\mathcal{V}})\}$.*

Proof. For every $q \in \mu_{\mathcal{V}},$ there is some $G \in {}^*\mathcal{V}$ such that $q \in G \subset \mu_{\mathcal{V}}$.

A nonempty collection $\mathcal{B}$ of subsets if X is a filter base, if $\emptyset \notin \mathcal{B}$ and if $A,\ B \in \mathcal{B}$, then there exists some $C \in \mathcal{B}$ such that $C \subset A \cap B$. A filter base is a filter subbase.

Theorem 6.4. *For a standard set X, let $\mathcal{B}$ be a standard filter base defined on X. If internal $A \subset \mu_{\mathcal{B}}$, then there exists some *-finite (and, hence, internal) $\Omega \subset {}^*\langle\mathcal{B}\rangle$ such that ${}^\sigma\langle\mathcal{B}\rangle \subset \Omega$ and $A \subset A_0 = \cap\{B \mid B \in \Omega\} \subset \mu_{\mathcal{B}}$, and $A_0 \in {}^*\langle\mathcal{B}\rangle$.*

Corollary 6.4.1. *For a standard set X, let $\mathcal{B}$ be a standard filter base defined on X. Then $\mu_{\mathcal{B}} = \cup\{G \mid (G \in {}^*\langle\mathcal{B}\rangle) \wedge (G \subset \mu_{\mathcal{B}})\}$.*

Theorem 6.5. *Given an internal set A and a standard filter subbase $\mathcal{F}$ such that $A \cap {}^*F \neq \emptyset$ for each $F \in \mathcal{F}$. Then $A \cap \mu_{\mathcal{F}} \neq \emptyset$.*

Proof. Consider the internal binary relation on $\mathcal{B} = \langle\mathcal{F}\rangle$.

$$\Phi = \{(a, b) \mid (b \in A) \wedge (b \in a) \wedge (a \in {}^*\mathcal{B})\}.$$

Suppose that $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subset \Phi$ and $a_i \in {}^\sigma\mathcal{B}$. Since $\mathcal{B}$ is closed under finite intersection, we have nonempty $a \in \mathcal{B}$, where ${}^*a = {}^*b_1 \cap \cdots {}^*b_n$. Thus there is some internal $b' \in {}^*a$ such that $(a_1, b'), \ldots, (a_n, b') \subset \Phi$. Since the cardinality of $\mathcal{B}$ less than κ, then Φ is, at least, concurrent on ${}^\sigma\mathcal{B}$ Thus there exists some $q \in A \cap {}^*B$ for each $B \in \mathcal{B}$. This implies that $A \cap \mu_{\mathcal{B}} = A \cap \mu_{\mathcal{F}} \neq \emptyset$ and the proof is complete.

Theorem 6.6. *Let $\mathcal{B}$ be a standard filter base and suppose that an internal set $\Lambda \subset {}^*\mathcal{B}$ has the property that ${}^\sigma\mathcal{B} \subset \Lambda$. Then there exists some internal $A \in \Lambda$ such that $A \subset \mu_{\mathcal{B}}$.*

Proof. Consider the internal (bounded) binary relation

$$\Phi = \{(b, a) \mid ((b \in {}^*\mathcal{B}) \wedge (a \in \Lambda) \wedge (a \subset b)\}.$$

Let $\{(b_1, a_1) \ldots, (b_n, a_n)\} \subset \Phi$ and $a_i{}^\sigma\mathcal{B}$. Then there is some $b' \in {}^\sigma\mathcal{B}$ such that $b' \subset b_1 \cap \cdots \cap b_n$. But $b' \in \Lambda$. Hence, $\{(b_1, b') \ldots, (b_n, b')\} \subset \Phi$. The κ-saturation, there exists some internal $A \in \Lambda$ such that $A \subset {}^*B$ for each $B \in \mathcal{B}$. Hence, $A \subset \mu_{\mathcal{B}}$ and the proof is complete.

Theorem 6.7 *Let $\mathcal{B}$ be a standard filter base and suppose that an internal set $\Lambda \subset {}^*\mathcal{B}$ has the property that if $G \in {}^*\mathcal{B}$ and $G \subset \mu_{\mathcal{B}}$, then $G \in \Lambda$. Then there exists some $B \in \mathcal{B}$ such that ${}^*B \in \Lambda$.*

Proof. Assume the hypothesis and that there is no $B \in \mathcal{B}$ such that ${}^*B \in \Lambda$. Then the internal set ${}^*\mathcal{B} - \Lambda \subset {}^*\mathcal{B}$ satisfies the hypothesis for the "Λ" of Theorem 6.6. Thus there exists some $A \in {}^*\mathcal{B} - \Lambda$ such that $A \subset \mu_{\mathcal{B}}$. But from the hypothesis of this theorem, such an $A \in \Lambda$. This contradiction complete the proof.

In order to obtain a significant result that characterizes S-continuity, we need the following additional fact.

Theorem 6.8 *Consider standard X. Let internal $\mathcal{B} \subset {}^*F({}^*X)$, where ${}^*F({}^*X)$ is the set of all *-finite subsets of ${}^*X$. Suppose that whenever $E \in {}^*F({}^*X)$ and ${}^\sigma X \subset E \subset {}^*X$ then $E \in \mathcal{B}$. Then there exists $F \in F(X)$ such that ${}^*F \in \mathcal{B}$.*

Proof. To establish this, let $\mathcal{F} = \{F \mid X - F \in F(X)\}$. Then $\mathcal{F}$ is a filter on X. This theorem is but an equivalent statement of theorem 6.6 in terms of the filter $\mathcal{F}$. Establishing this fact completes the proof.

We are now able to properly characterize the concept of S-continuity relative to standard pseudo-metric generated spaces.

Theorem 6.9. *For standard X, Y, consider the pseudo-metric generated spaces $({}^*X, {}^\sigma\Lambda)$, $({}^*Y, {}^\sigma\Pi)$. An internal $f\colon {}^*X \to {}^*Y$ is S-continuous at $q \in {}^*X$ if and only if for each $\epsilon \in \mathbb{R}^+$ and each ${}^*\pi \in {}^\sigma\Pi$ there exists a finite set ${}^*\lambda_i$, $1 \leq i \leq n$ and positive $\delta \in \mathbb{R}^+$ such that whenever $x \in {}^*X$ and ${}^*\lambda_i(x,q) < \delta$ for each i, $1 \leq i \leq n$, then ${}^*\pi(f(x), f(q)) < \epsilon$.*

Proof. $\Rightarrow$ First, suppose that f is S-continuous at $q \in {}^*X$, ${}^*\pi \in {}^\sigma\Pi$ and let $\epsilon \in \mathbb{R}^+$. For any internal binary relation A, let $D(A)$ denote the internal domain and $R(A)$ the internal range. Consider the internal set

$$T(\epsilon) = \{K \mid (\emptyset \notin D(K)) \wedge (\emptyset \notin R(K)) \wedge (K \in {}^*F({}^*\Lambda \times {}^*\mathbb{R}^+) \wedge (\forall\lambda\forall\delta\forall x$$

$$((\lambda \in D(K)) \wedge (\delta \in R(K)) \wedge (x \in {}^*X) \wedge (\lambda(x,q) < \delta) \to$$

$$({}^*\pi(f(x), f(q)) < \epsilon)))\}.$$

By κ-saturation, we know that there exists a *-finite $K_0 \subset {}^*\Lambda \times {}^*\mathbb{R}^+$ such that ${}^\sigma(\Lambda \times \mathbb{R}^+) \subset K_0$. Thus $\emptyset \neq T(\epsilon)$. Further, suppose that $K_1 \in {}^*F({}^*\Lambda \times {}^*\mathbb{R}^+)$ and ${}^\sigma\Lambda \times \mathbb{R}^+ \subset K_1$. Then $D(K_1) = G_1 \in {}^*F({}^*\Lambda)$, ${}^\sigma\Lambda \subset G_1$ and $R(K) = H_1 \in {}^*F(\mathbb{R}^+)$, $\mathbb{R}^+ \subset H_1$. Now $x \approx q$ implies that ${}^*\lambda(x,q) < \delta$ for each ${}^*\lambda \in {}^\sigma\Lambda$ and each $\delta \in \mathbb{R}^+$. But, S-continuity implies that ${}^*\pi(x,q) < \epsilon$. Hence $K_1 \in T(\epsilon)$. Since $T(\epsilon)$ is internal, then Theorem 6.8 implies that there exists standard K' such that $K' \in F(\Lambda \times \mathbb{R}^+)$ and such that ${}^*K' \in T(\epsilon)$. Consequently, there is positive $n \in \mathbb{N}$ and $\lambda_i \in D(K') \subset \Lambda$ when $1 \leq i \leq n$ and a positive m and $\delta_j \in R(K') \subset \mathbb{R}^+$ when $1 \leq j \leq m$ and for each $x \in X$, if for each i

and for each j ${}^*\lambda_i(x,q) < \delta_j$, then ${}^*\pi(f(x), f(q) < \epsilon$. Now simply consider $\delta = \min\{\delta_1, \ldots, \delta_m\}$ and $\Rightarrow$ holds.

$\Leftarrow$ Suppose that f is not S-continuous at q. Then there exists some $x_0 \in \mu_{{}^\sigma\Lambda}(q) = \cap\{\mu_{{}^*\pi}(q) \mid {}^*\pi \in {}^\sigma\Pi\}$, such that $f(x_0) \notin \mu_{{}^*\Pi}(f(q))$. Thus there is some $\pi \in \Pi$ such that $f(x_0) \notin \mu_\pi(f(q))$. Let standard $\epsilon = \min\{1, \mathtt{st}(\pi(f(x_0), f(q)))/2\}$ if $\pi(f(x_0), f(q)) \in \mathcal{O}$, in which case $\epsilon > 0$ since $\mathtt{st}(\pi(f(x_0), f(q))/2 \neq 0$. Otherwise take $\epsilon = 1$. Now for all standard $\delta > 0$, we have that for each ${}^*\lambda \in {}^\sigma\Lambda$, $\lambda(x_0, q) < \delta$, but ${}^*\pi(f(x_0), f(q)) \geq \epsilon$. The proof is complete.

Corollary 6.9.1. *For standard X, Y, consider the pseudo-metric generated spaces $({}^*X, {}^\sigma\Lambda)$, $({}^*Y, {}^\sigma\Pi)$. An extended standard map ${}^*f\colon {}^*X \to {}^*Y$ is S-continuous at ${}^*p \in {}^\sigma X$ if and only if $f\colon (X, \Lambda) \to (Y, \Pi)$ is at continuous at $p \in X$.*

Notice that the δs and ϵs that appear on Theorem 6.9 are standard real numbers. This theorem shows that close relationship between the concept of S-continuity and the concept of continuity. For the only difference within our κ saturated model between these two concepts when viewed from the external nonstandard physical world is that one hand the xs are members of ${}^*X$ while on the other they elements of X.

One of the major facts about S-continuous functions is found in Theorem 1.1 in [4, p. 805.] As pointed out, for this theorem, the topological space X need not be compact and the first two parts of the theorem hold. In this theorem, the term microcontinuous is equivalent to S-continuous. For simplicity of notation, the sets X, Y are considered as a subset of the ground set that is used to generate our ultraproduct structure.

Theorem 6.10. *Suppose that you have the topological spaces X, Y, Y regular Hausdorff, where topological monads are defined at standard points, and an internal $f\colon {}^*X \to {}^*Y$. At $p \in X$, let f be S-continuous and suppose that there exists some $r \in Y$ such that $f({}^*p) \approx r$. Then any function $F\colon X \to Y$ such that $F(p) = \mathtt{st}(f({}^*p))$ is continuous at p in the topological sense. Further, if $q \approx {}^*p$, then $f(q) \approx {}^*F({}^*p)$.* [Note. The standard part operator as defined in this theorem, brings points all the way back to the original standard set Y. It is not significant that this operator can be considered as defined on $Z \subset {}^*Y$ and $\mathtt{st}(Z) \subset {}^\sigma Y$.]

Theorem 6.10 shows, in all generality, how S-continuity in general topological spaces leads directly to a standard function because the internal function ignores infinitesimal discontinuities. If the space Y is not be Hausdorff, then, by the Axiom of Choice, continuous function(s) can still be constructed.

Definition 6.8 (S-convergence) Let internal sequence $s\colon {}^*\mathbb{N} \to (X, \Lambda)$, where (X, Λ) is a pseudo-metric generated space. Then s S-converges to $q \in X$ if $s_\omega \in \mu_\Lambda(q)$ for each infinite $\omega \in \mathbb{N}_\infty$.

Theorem 6.11. *If, for pseudo-metric generated spaces, internal $f: (X, \Lambda) \to (Y, \Pi)$ is S-continuous at $q \in X$ and the internal sequence $s: {}^*\mathbb{N} \to (X, \Pi)$ S-converges to q, then the internal sequence $f(s)$ S-converges to $f(q)$.*

Proof. Suppose that the internal sequence s S-converges to $q \in X$. Let $\omega \in \mathbb{N}_\infty$. Then $s_\omega \in \mu_\Lambda(q)$. Thus, $f(s_\omega) = (fs)_\omega \in \mu_\Pi(f(q))$ and the proof is complete.

The next theorem is similar to Theorem 6.9, relates S-convergence to standard approximations as well as to standard sequences.

Theorem 6.12. *For a pseudo-metric generated space (X, Λ), an internal sequence $s: {}^*\mathbb{N} \to X$, S-converges to $q \in X$ if and only if for each positive $\epsilon \in \mathbb{R}$ each $\lambda \in \Lambda$ there exists some $M \in \mathbb{N}$ such that for each $m > M$ (in ${}^*\mathbb{N}$), it follows that $\lambda(s_m, q) < \epsilon$.*

Proof. ($\Rightarrow$) Suppose the internal sequence s is S-convergent to $q \in X$. Let $0 < \epsilon \in \mathbb{R}$. Consider the internal set

$$m(\epsilon) = \{m \mid (m \in {}^*\mathbb{N}) \land (\forall n((n \in {}^*\mathbb{N}) \land (n > m) \to \lambda(s_n, q) < \epsilon)))\}.$$

From the definition of S-convergence $\mathbb{N}_\infty \subset m(\epsilon)$. However, the set $m(\epsilon)$ has an internal range. Hence, there exists some standard $m \in \mathbb{N}$ such that $m \in m(\epsilon)$. Consequently, the conclusion follows.

($\Leftarrow$) Suppose that s is not S-convergent to q. Then there exists some $s_\omega,\ \omega \in \mathbb{N}_\infty$ such that $s_\omega \notin \mu_\Lambda(q)$. Hence, there is some $\lambda \in \Lambda$ such that $s_\omega \notin \mu_\lambda(q)$. Let standard $\epsilon = \min\{1, \mathtt{st}(\lambda(s_\omega, q))/2\}$ if $\lambda(s_\omega, q) \in \mathcal{O}$, in which case $\epsilon > 0$ since $\mathtt{st}(\lambda(s_\omega, q)/2) \neq 0$. Otherwise take $\epsilon = 1$. Thus there exists some $\omega \in \mathbb{N}_\infty$ such that and $\lambda(s_\omega, q) \geq \epsilon$. The proof is complete.

Corollary 6.12.1. *For a standard pseudo-metric space (X, ρ_X), a standard sequence $s: \mathbb{N} \to (X, \rho_X)$ is convergent to $p \in X$ if and only if ${}^*s$ is S-convergent to p.*

Now we need to define the limited points for internal pseudo-metrics. For an internal pseudo-metric ρ_X, you have a set of limited points per $p \in X$.

Definition 6.9 (Limited Points for Pseudo-metric Generated Spaces) Let (X, Λ) be a pseudo-metric generated space. Let $q \in X$ Then $\mathcal{O}_\Lambda(q) = \{x \mid (x \in X) \land \forall \lambda((\lambda \in \Lambda) \to \lambda(x, q) \in \mathcal{O})\}$.

Theorem 6.13. *Let $({}^*X, \Lambda)$ be a pseudo-metric generated space. Suppose that each $\lambda \in \Lambda$ is determined by an internal pseudo-norm $\|\cdot\|_\lambda$ and that ${}^*X$ is an internal linear space over ${}^*\mathbb{R}$. Suppose that for each $p \in X$ and each $\|\cdot\|_\lambda$, $\|p\|_\lambda \in \mathcal{O}$. Then for each $p \in X$, $\mathcal{O}_\Lambda(p) = \mathcal{O}_\Lambda(\mathbf{0}) = \{x \mid (x \in {}^*X) \land \forall \lambda((\lambda \in \Lambda) \to (\|x\|_\lambda \in \mathcal{O}\} = \mathcal{O}_X$.*

Proof. Let $x \in \mathcal{O}_\Lambda(p)$ for $p \in X$. Then for any $\lambda \in \Lambda$ it follows that $|\ \|x\|_\lambda - \|p\|_\lambda\ | \in \mathcal{O}$. Hence, $\|x\|_\lambda \in \mathcal{O}$ since $\|p\|_\lambda \in \mathcal{O}$.

Conversely, let $x \in \mathcal{O}_\Lambda(\mathbf{0})$ and $p \in X$. Then for each $\lambda \in \Lambda$, $\|x\|_\lambda \in \mathcal{O}$. But $\|p\|_\lambda \in \mathcal{O}$ implies, since $\|x - p\|_\lambda \leq \|x\|_\lambda + \|p\|_\lambda \in \mathcal{O}$, that $x \in \mathcal{O}_\Lambda(p)$. The proof is complete.

Corollary 6.6.1. *Let the standard pseudo-metric ρ_X be defined on $X \times X$ and be generated by a standard semi-norm $\|\cdot\|_X$. If $r,\ p \in X$, then $\mathcal{O}_\rho(p) = \mathcal{O}_\rho(r) = \mathcal{O}_X$.*

By Robinson's Theorem 4.3, the set T can be replaced by the set ${}^*C^\infty \cap T$. Hence, the usual practice has been to consider defining sets of internal semi-norms on the set ${}^*C^\infty$ and consider the restriction such a set of internal semi-norms to the test space D. This collection can be composed of the nonstandard extensions of the customary set of standard semi-norms so that they correspond to the concept of Schwartz generalized functions and other standard types of generalized functions. However, it is also possible to broaden the collection of internal semi-norms in various ways. This is done in section 10.4 of reference [8].

7. Per-generalized Functions and S-continuity.

Theorem 7.1. *Let generalized function $f \in T$ and standard $p \in \mathbb{R}$. Suppose that f is S-continuous at p. Then $f(p) \in \mathcal{O}$.*

Proof. Let f be S-continuous at $p \in \mathbb{R}$. Assume that $f(p)$ is not limited. Without loss of generality, assume that $f(p)$ is a positive infinite hyperreal number. Now for each $x \approx p$, it follows that $f(x) > (1/2)f(p)$ since by S-continuity $f(x) \approx f(p)$. Note: For an infinite Λ and any $\epsilon \in \mu(0)$, the infinite $\Lambda + \epsilon > (1/2)\Lambda$. By the internal definition method, define that internal set

$$D = \{x \mid (x > 0) \land (x \in {}^*\mathbb{R}) \land (|x - p| > 0 \to (f(x - p) > (1/2)f(p)))\}.$$

Since D contains all of the positive infinitesimals, then by a modified 10.1.1 in [3], D contains a standard positive a. Consider the standard interval $[p - a, p + a]$. It is not difficult to construct a non-negative $h \in \mathcal{D}$ such that $h(x) = 1$, for each x such that $|x| \leq (1/2)a$ and $h(x) = 0$ for $|x| \geq a$. Now let $g = h(x - p)$. Since $g \in \mathcal{D}$,

$$ {}^*\!\int_{-\infty}^{\infty} f\, {}^*g = {}^*\!\int_{p-a}^{p+a} f\, {}^*g > (1/2)af(p).$$

The results follows from this contradiction.

Corollary 7.1.1. *Let internal $f \in T$ and standard $p \in \mathbb{R}$. Suppose that f is S-continuous at p. Then any standard function $F\colon \mathbb{R} \to \mathbb{R}$ such that $F(p) = \mathtt{st}(f(p))$ is continuous at p and $f(\mu(p)) \subset \mu(F(p))$.*

Theorem 7.2. *If $f \in T$ is S-continuous at each $p \in \mathbb{R}$, then $F(p) = \mathtt{st}(f(p))$ is continuous on $\mathbb{R}$.*

{**Remark.** If the function $f \in T$ satisfies the requirements in Theorem 7.2, then $f(\mathcal{O}) \subset \mathcal{O}$. If f is also a surjection, then $f(\mathcal{O}) = \mathcal{O}$.}

Example 7.1 We show that the converse of Theorem 7.2 does not hold. First, consider for any given standard $a > 0$, the set $A = [-a, 0) \cup (0, a]$, and the standard function defined by

$$f(x) = \begin{cases} 1, & x \in A \\ 0, & x \in \mathbb{R} - A. \end{cases}$$

Then f^2 is Riemann integrable on any real $[c, d],\ c \leq d$ and since $\sup\{f^2(x) \mid x \in \mathbb{R}\} = 1,\ \int_c^d f^2 \leq (d - c)$. It follows from this that $|\int_{-\infty}^{\infty} fg| \leq |\int_{-h}^{h} g| \in \mathbb{R}$ for some $h \in \mathbb{R}$. Now consider the internal function k obtained by means of the definition for f but let positive $a = \epsilon \in \mu(0)$. Selecting $c,\ d \in \mathcal{O}$, it follows that $k \in T$. Notice that for each $p \in \mathbb{R},\ k(p) = 0$. Thus $K(p) = \mathtt{st}(k(p)) = 0$ is a continuous function on $\mathbb{R}$. Let $x = \epsilon/2 \in \mu(0)$. Then $k(x) = 1 \not\approx 0$. Thus k is not S-continuous at $x = 0$.

One of the major advantages in using the nonstandard equivalence class method is that various members of a pre-generalized function α can be specifically analyzed. This is not the case for the standard approaches to this subject.

Theorem 7.3. *If $f \in \alpha$ is S-continuous at each $p \in \mathbb{R}$, then, for the continuous function F defined by $F(p) = \mathtt{st}(f(p)),\ {}^*F \in \alpha$.*

Proof. Since ${}^*F \in T$, it follows that ${}^*F \in \alpha$ if and only if ${}^*\!\int_{-\infty}^{\infty}(f - {}^*F)\,{}^*g \approx 0$ for all $g \in \mathcal{D}$. From the definition of g, there is $c \in \mathbb{R}$ such that

$${}^*\!\int_{-\infty}^{\infty}(f - {}^*F)\,{}^*g = {}^*\!\int_{-c}^{c}(f - {}^*F)\,{}^*g.$$

Since $[-c, c]$ is compact, if $x \in {}^*[-c, c]$, then there exists some $p \in [-c, c]$ such that $x \approx p$. From S-continuity, standard part operator properties, and continuity, $f(x) \approx f(p) \approx F(p) \approx {}^*F(x)$ implies that $f(x) \approx {}^*F(x)$. Consequently, $|f(x) - {}^*F(x)| \approx 0$. Consider the internal set $A = \{y \mid (y \in {}^*\mathbb{R}) \land \exists x((x \in {}^*[-c, c]) \land (y = |f(x) - {}^*F(x)|))\}$. Since $[-c, c]$ is compact, $F([-c, c]) = [a, b]$ implies that ${}^*F({}^*[-c, c]) = {}^*[a, b]$. Further, from above, $f({}^*[-c, c]) \subset [-c - 1, c + 1]$. Consequently, the internal set A is *-bounded. Recalling that the ${}^*\sup = \sup$, it follows by *-transfer, that $\sup A \in {}^*\mathbb{R}$. But for each positive $r \in \mathbb{R}$ and each $y \in A$, $0 \leq y < r$. Hence, the $\sup A$ is not a positive real number. Again by *-transfer and the fact that A is internal, $\sup A \in {}^*\mathbb{R}$. Thus $\sup A \approx 0$. But,

$$\left|{}^*\!\int_{-c}^{c}(f - {}^*F)\,{}^*g\right| \leq \sup\{|f(x) - {}^*F(x)| \mid x \in {}^*[-c, c]\}\left(\int_{-c}^{c}|g|\right).$$

Hence, ${}^*\!\int_{-c}^{c}(f - {}^*F)\,{}^*g \approx 0$ and the proof is complete.

Corollary 7.3.1. *If $f \in \alpha$ is S-continuous at each $p \in \mathbb{R}$, then α is a $\mathcal{D}'$-pre-generalized function.*

Proof. The *-continuous function ${}^*F \in \alpha$. Consider a sequence $\{g_n\} \subset \mathcal{D}$ such that $g_n(x) = 0$ for all n and for all x such that $|x| > c \in \mathbb{R}$. Further suppose that $g_n \to 0$ uniformly on for all x such that $|x| \leq c$. Then $\lim_{n\to\infty} {}^*F[g_n] = \mathtt{st}({}^*(\lim_{n\to\infty}\langle F, g_n\rangle)) = \mathtt{st}({}^*\langle F, \lim_{n\to\infty} g_n\rangle)) = \mathtt{st}(0) = 0$.

The existence of a function f in a pre-generalized function that is S-continuous at various members of ${}^*\mathbb{R}$ seems to be of some significance. Indeed, the standard part of members of a pre-generalized function that are S-continuous at the same point in $\mathbb{R}$ cannot be distinguished one from another at that point.

Theorem 7.4. *Suppose that $f, h \in \alpha$ and that f and h are S-continuous at $p \in \mathbb{R}$. Then $\mathtt{st}(f(p)) = \mathtt{st}(h(p))$ and if $x \approx p$, then $f(x) \approx h(x)$.*

Proof. We know from Theorem 6.1 that $f(p),\ h(p) \in \mathcal{O}$. Thus $\mathtt{st}(f(p)),\ \mathtt{st}(h(p)) \in \mathbb{R}$. All we need to do is to show that $f(p) \approx h(p)$. Since $f,\ h \in \alpha$, the function $k = f - h \in T_0$. Suppose that $f(p) - h(p)$ is not infinitesimal. Without loss of generality, in this case, consider $k(p) = f(p) - h(p) > r > 0,\ r \in {}^*\mathbb{R} - \mu(0)$. As in the proof of Theorem 7.1, there is a $g \in \mathcal{D}$, such that $k[g]$ is not infinitesimal; a contradiction. Thus $f(p) \approx h(p)$. Obviously, if $x \in p$, S-continuity implies that if $x \approx p$, then $f(x) \approx f(p) \approx h(p) \approx h(x)$. Hence $f(x) \approx h(x)$ and the proof is complete.

Due to Theorem 7.4, a pre-generalized function α that contains an internal function f that is S-continuous at $p \in \mathbb{R}$ can be considered a function itself.

Definition 7.1. (α as a Function on $\mathbb{R}$.) If $f \in \alpha$ and f is S-continuous at $p \in \mathbb{R}$ let $\alpha(p) = \mathtt{st}(f(p)) = F(p)$.

Theorem 7.4. shows that definition 7.1 leads to a function-like pre-generalized function $\alpha(p)$. However, one other aspect of this α function concept needs to be addressed. How unique is such a α function with respect to members of T?

Theorem 7.5. *Suppose that $f \in \alpha$ is S-continuous at each $p \in \mathbb{R}$. Let $h \in T$ be S-continuous at each $p \in \mathbb{R}$ and $h(p) = f(p)$. Then $h \in \alpha$.*

Proof. (Notice that simply because $h(p) = f(p)$ at the standard points does not imply the functions are equal at the nonstandard points.) What is needed is to show that $\langle (f-h), {}^*g \rangle \approx 0$ for each $g \in \mathcal{D}$. Consider $f - h \in T$. Let $g \in \mathcal{D}$. Then there exists some positive $c \in \mathbb{R}$ such that $g(x) = 0$ for $|x| \geq c$. Since $f(\mu(a)) \subset \mu(f(p))$ and $h(\mu(a)) \subset \mu(h(a))$ and $f(a) = h(a),\ (f-h)(\mu(a)) \subset \mu(0)$. Consider a specific *standard* positive ϵ and the internal set $D(\epsilon) =$

$$\{y \mid (y \in {}^*\mathbb{R}) \wedge \forall x((x \in {}^*\mathbb{R}) \wedge (|x-a| < \epsilon) \rightarrow (|f(x) - h(x)| < y))\}.$$

Then $D(\epsilon)$ contains all of the positive infinitesimals. As in the proof of Theorem 7.1, there exists a standard positive η_a such that such that $|f(x) - h(x)| < \epsilon$ for each x such that $|x - a| < \eta_a$. For a fixed ϵ, for each $a \in [-c, c]$, the positive standard η_a exists. The set of open intervals $\{U_a \mid (a \in [-c, c]) \wedge (U_a = \{x \mid (a - \eta_a < x < a + \eta_a\})\}$ is an open cover of compact $[-c, c]$ and as such there exists a finite subset $\{U_1, \ldots, U_n\}$ that covers $[-c, c]$. Also, $|f(x) - h(x)| < \epsilon$ for each $x \in {}^*U_i$. Since the set $\{U_i\}$ is nonempty and finite it can be adjusted by moving in the end points if necessary (remove some of the open intervals) so that the shortened open intervals thus obtained have the property that no point in $[-c, c]$ is contained in more than two such intervals and such that the sum of the lengths is not greater than say $4c$. Denote these adjusted intervals by H_i. Again $|f(x) - h(x)| < \epsilon$ for each $x \in {}^*H_i$. This same procedure can be done for any arbitrary positive ϵ.

Now let $\lambda = \max |g(x)|$. It is a known fact that $g(x) = g_1(x) + \cdots + g_n(x)$, where each $g_i \in \mathcal{D}$ and, for $i = 1, \ldots, n$, $g_i(x) = 0, x \in \mathbb{R} - H_i$ and $|g_i(x)| \leq \lambda,\ x \in \mathbb{R}$. Hence

$$\left| {}^*\!\!\int_{-\infty}^{\infty} (f - h)\, {}^*g \right| = \left| \sum_{i=1}^{n} {}^*\!\!\int_{H_i} (f - h)\, {}^*g_i \right| \leq$$

$$\sum_{i=0}^{n} {}^*\!\!\int_{H_i} \epsilon\lambda \leq 4c\epsilon\lambda.$$

But, as pointed out, positive ϵ is arbitrary. Thus

$$\left| {}^*\!\!\int_{-\infty}^{\infty} (f - h)\, {}^*g \right| \approx 0$$

and the proof is complete.

Corollary 7.5.1. *Suppose that $f,\ h \in T$ are S-continuous at each $p \in \mathbb{R}$ and $f(p) = h(p)$ for $p \in \mathbb{R}$. Then there is a unique α such that $f,\ h,\ {}^*F \in \alpha$, $F = H$, F is continuous on $\mathbb{R}$ and $\alpha(p) = \mathtt{st}(f(p)) = \mathtt{st}(h(p)) = F(p)$ for each $p \in \mathbb{R}$.*

Another aspect of this idea of S-continuity and pre-generalized functions is the observation that defining $\alpha(p) = \mathtt{st}(f(p))$, for some $f \in \alpha$ that is S-continuous at $x = p$, is independent of the S-continuous function contained in α by Theorem 7.4. Thus many α can be considered as functions on large domains of real numbers. For example, the function d, in example 4.1, is in T and is S-continuous at every nonzero real number. Thus using this function the Dirac $\mathcal{D}'$-pre-generalized function δ is a function $\delta(x)$ for all nonzero real numbers. Then we have certain algebraic properties associated with the $\alpha \in T/T_0$ that are functions at certain standard points.

Definition 7.2 (α as a Function) Call a per-generalized function α a function at $p \in \mathbb{R}$, with value $\alpha(p)$, if there is an $f \in \alpha$ that is S-continuous at p and let $\alpha(p) = \mathtt{st}(f(p))$.

Theorem 7.6. *If $\alpha,\ \beta$ are functions at p, then $\gamma = \alpha \pm \beta$ is a function at p and $\gamma(p) = \alpha(p) \pm \beta(p)$.*

Proof. From the hypothesis, there exist two internal $f,\ h \in T$ such that $f,\ h$ are S-continuous at p and $f \in \alpha,\ h \in \beta$. Hence, the internal function $f \pm h \in T$ is S-continuous at p and $f \pm h \in \gamma$. Since $\mathtt{st}(f(p) \pm h(p)) = \mathtt{st}(f(p)) + \mathtt{st}(h(p))$, it follows that $\gamma(p) = \alpha(p) \pm \beta(p)$ and this completes the proof.

From Theorem 4.3, we know that every member of $\mathcal{F}$ is generated by a member of ${}^*C^{\infty}$. This leads to the concept of the derivative of a generalized function. Unfortunately, if you want to define multiplication for the functions in T and use the usual concept that multiplication is independent of the member chosen from one or both of two pre-generalized functions α and β, then multiplication must be restricted to certain per-generalized functions. Theorem 7.4 states that if ${}^*f,\ {}^*h \in {}^\sigma C^\infty$ and ${}^*f,\ {}^*h \in \alpha$, then, since ${}^*f,\ {}^*h$ are S-continuous at each $p \in \mathbb{R}$, $\mathtt{st}({}^*f(p)) = f(p) = \mathtt{st}({}^*h(p)) = h(p)$. Thus $f = h$. Whenever possible such a unique member of ${}^\sigma C^\infty$ can be used to generate a unique product-like per-generalized function.

Theorem 7.7. *Let ${}^*f \in {}^\sigma C^\infty$ and ${}^*f \in \alpha$ (hence, a $\mathcal{D}'$-pre-generalized function). Suppose that $h,\ k \in \beta$. Then there exists a pre-generalized function γ such that $({}^*f)h,\ ({}^*f)k \in \gamma$.*

Proof. Notice that if ${}^*f \in {}^\sigma C^\infty$ and ${}^*g \in {}^\sigma\mathcal{D}$, then ${}^*f\,{}^*g \in {}^\sigma\mathcal{D}$. From Theorem 3.1 (c), (e), $({}^*f)h, ({}^*f)k \in T$. Thus there is a pre-generalized function γ such that $({}^*f)h \in \gamma$. Consider $\langle ({}^*f)h - ({}^*f)k,\ {}^*g\rangle$ for any $g \in \mathcal{D}$. Then $\langle ({}^*f)h - ({}^*f)k,\ {}^*g\rangle = \langle h - k, ({}^*f)\,{}^*g\rangle \approx 0$.

Definition 7.3. (Customary Products) Let ${}^*f \in {}^\sigma C^\infty,\ {}^*f \in \alpha$. Then the pre-generalized function γ such that for each $h \in \beta,\ ({}^*f)h \in \gamma$ is the customary product of α and β and is denoted by $\gamma = \alpha\beta$.

Theorem 7.8. *Let $\{{}^*f, {}^*h\} \subset {}^\sigma C^\infty$ and ${}^*f \in \alpha,\ {}^*h \in \beta$. Then for $\gamma = \alpha\beta$ it follows that $\gamma' = \alpha'\beta + \alpha\beta'$.*

Proof. We know that $({}^*f\,{}^*h)' \in \gamma'\ {}^*f' \in \alpha',\ {}^*h' \in \beta'$. Of course, $({}^*f\,{}^*h)' = ({}^*f')\,{}^*h + ({}^*f)\,{}^*h'$. Notice that $({}^*f')h \in \alpha'\beta,\ ({}^*f)\,{}^*h' \in \alpha\beta'$. Hence, $({}^*f')\,{}^*h + ({}^*f)\,{}^*h' \in \alpha'\beta + \alpha\beta'$. Then we have that $(({}^*f)\,{}^*h)' \in \gamma'$. But the pre-generalized functions are equivalence classes. Thus $\gamma' = \alpha'\beta + \alpha\beta'$ and this completes the proof.

Lemma 7.1. *Let $f: {}^*\mathbb{R} \to {}^*\mathbb{R}$ and $h: {}^*\mathbb{R} \to {}^*\mathbb{R}$ be S-continuous at $p \in \mathbb{R}$. Then the product function $k(x) = f(x)h(x)$ is S-continuous at p.*

Proof. Let $q \approx p$. Then $k(q) - k(p) = f(q)h(q) - f(p)(h(p) = f(q)(h(q) - h(p)) + (f(q) - f(p))h(p)$. Notice that $f(q), h(p) \in \mathcal{O}$ and $(h(q) - h(p)), (f(q) - f(p)) \in \mu(0)$. Hence, $k(q) - k(p) \in \mu(0)$. The result is complete.

Theorem 7.9. *Let ${}^*f \in {}^\sigma C^\infty$ and ${}^*f \in \alpha$. Assume that β is a function at $p \in \mathbb{R}$. Let $\gamma = \alpha\beta$. Then γ is a function at $p \in \mathbb{R}$ and $\gamma(p) = \alpha(p)\beta(p)$.*

Proof. Since f is S-continuous at p, $f(p) = \alpha(p)$. Further, there is some $h \in \beta$ that is S-continuous at p and $\beta(p) = \mathtt{st}(h(p))$. Consider $k = fh \in \gamma$. From Lemma 7.1, γ is a function at p and since $\mathtt{st}(f(p)h(p)) = \mathtt{st}(f(p))\mathtt{st}(h(p))$ the result follows.

Another result similar to Theorem 7.5, but not as definitive, can be obtained using the Theorem 7.5 method.

Theorem 7.10. *Let $f \in T$ be S-continuous at each x such that $a \leq x \leq b,\ a < b,\ a, b \in \mathbb{R}$ and α a function at each x such that $a \leq x \leq b,\ a < b,\ a, b \in \mathbb{R}$. Then for every $g \in \mathcal{D}$ such that the support of g is a subset of $[a, b]$, and every $h \in \alpha$ it follows that $f[g] = h[g]$.*

Proof. Let $f \in \beta$. What is needed is to show that $\langle (f - h), {}^*g \rangle \approx 0$ for each $g \in \mathcal{D}$ such that the support of g is a subset of $[a, b]$. Taking the c in the proof of Theorem 7.5 such that $[-c, c] \subset [a, b]$, the proof is exactly the same as that of Theorem 7.5.

Any pre-generalized function that is a function on $\mathbb{R}$ is an $\mathcal{D}'$-per-generalized function. This idea can be extended to the k derivatives of per-generalized functions.

Theorem 7.11. *Suppose that α is a function on $\mathbb{R}$. Then for each positive $k \in \mathbb{N}$, the pre-generalized function $\alpha^{(k)}$ is a $\mathcal{D}'$-pre-generalized function.*

Proof. From the hypothesis, there is a function F continuous on $\mathbb{R}$ such that ${}^*F \in \alpha$. From Theorem 4.3, there is some $h \in {}^*C^\infty$ such that $h \in \alpha$ and, from the Definition of $\alpha^{(k)}$, $h^k \in \alpha^{(k)}$. Let $\{g_n\}$ be a sequence of members of $\mathcal{D}$ such that $g_n(x) = 0$ for all n and all x such that $|x| > c,\ c > 0$ and $\{g_n^{(k)}\}$ converges uniformly in x for all positive k. All that is needed is to show that the sequence $h^{(k)}[g_n]$ converges to zero. We know by parts integration that $h^{(k)}[g_n] = (-1)^k h[g_n^{(k)}]$. Further, ${}^*F[g_n^{(k)}] = h[g_n^{(k)}]$. From uniform convergence, $\lim_{n\to\infty} {}^*F[g_n^{(k)}] = \lim_{n\to\infty} h[g_n^{(k)}] = \lim_{n\to\infty} h^{(k)}[g_n] = 0$. The result follows.

8. Generalizations and Beyond

Every theorem in the previous sections relative to pre-generalized functions, holds true if the domain ${}^*\mathbb{R}$ is changed to an open domain $\Omega \subset {}^*\mathbb{R}^m$. [Note: Theorem 4.3 still holds in the sense that there exists $f \in {}^*C^\infty(\mathbb{R}^m)$ that generates each pre-generalized function.] Depending upon the level of pre-generalized function differentiation desired, the test space $\mathcal{D}$ can be enlarged or reduced. For example, if you only wish that for each α, there exists $\alpha^{(k)}$ where $1 \leq k \leq m$. Then the test spaces can be the set of all functions $g\colon \mathbb{R} \to \mathbb{R}$ that have bounded support and are continuously differentiable of order m. In this case, all the previous theorems on pre-generalized functions, modified for this degree of differentiability, also hold. Then another approach that has proved to be accessible to nonstandard methods is the generalization due to Colombeau [8]. I mention that the new approach by Egorov [9] could lead to a very accessible nonstandard theory and needs to be fully explored.

Using section 6 and an appropriate collection of internal semi-norms, the concepts of the S-limit and S-convergence, and the like, can now be applied to pre-generalized functions.

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Mathematics Dept.
United States Naval Academy,
572C Holloway Rd,
Annapolis, MD 21402-5002.